Symmetries of κ Minkowski space-time and emergence of a curved momentum space

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- In special and general relativity simultaneity is relative but locality is absolute. This follows from the assumption that spacetime is a universal entity in which all of physics unfolds.
- However, all approaches to the study of the quantum-gravity problem suggest that locality must be weakened and that the concept of spacetime is only emergent and should be replaced by something more fundamental.
- A natural and pressing question is **whether it is possible to relax the universal locality** assumption in a controlled manner, such that it gives us a stepping stone toward the theory of quantum gravity?

- Planck length, $l_p = \sqrt{\hbar G}$, sets an absolute limit to how precisely an event can be localized, $\Delta x \sim l_p$. However, the Planck length is non zero only if G and $\hbar$ are non zero, so this hypothesis requires a full fledged quantum gravity theory.

- As an alternative, we can explore a "classical-non gravitational" regime of quantum gravity which still captures some of the key delocalising features of quantum gravity. In this regime, $\hbar$ and G are both neglected, while their ratio is held fixed:

$$\hbar, G \to 0; \qquad \sqrt{\frac{\hbar}{G}} \sim m_P$$

Mass scale m_P parameterizes non-linearities in momentum space. Remarkably, these non linearities can be understood as introducing a non trivial geometry on momentum space .

G. Amelino Camelia, Phys.Lett.B **510**, 2001.
J. Magueijo, L. Smolin, Phys. Rev. Lett **88** 190403,2002.
J.K. Glikman, Lect. Notes. Phys. **669**, 2005.

$$[\hat{X}^\mu, \hat{X}^\nu] = i(a^\mu \hat{X}^\nu - a^\nu \hat{X}^\mu) \qquad (1)$$

a^μ is a set of four real numbers (Lorentz scalars)

$$|a| = \sqrt{\eta_{\mu\nu} a^\mu a^\nu} = \frac{1}{\kappa} \sim L_p = [L]$$

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The above algebra is clearly not covariant under infinitesimal ISO(3,1) transformation

$$\hat{X}^\mu \rightarrow \hat{X'}^\mu = \hat{X}^\mu + \epsilon^\mu$$

$$\hat{X}^\mu \rightarrow \hat{X'}^\mu = \hat{X}^\mu + \omega^{\alpha\mu} \hat{X}_\alpha$$

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So to define the symmetry of this Kappa deformed space-time, one needs to deform the transformation rules.

Let us first look into the Lie algebra (iso(1,3)) of the usual Poincare generators:

$$
\begin{aligned}
[\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}] &= i(\eta_{\nu\rho}\hat{M}_{\mu\sigma} + \eta_{\mu\sigma}\hat{M}_{\nu\rho} - \eta_{\mu\rho}\hat{M}_{\nu\sigma} - \eta_{\nu\sigma}\hat{M}_{\mu\rho}) \\
[\hat{M}_{\mu\nu}, \hat{P}_{\rho}] &= i(\eta_{\nu\rho}\hat{P}_{\mu} - \eta_{\mu\rho}\hat{P}_{\nu}) \\
[\hat{P}_{\mu}, \hat{P}_{\nu}] &= 0
\end{aligned} \tag{2}
$$

where $\hat{M}_{\mu\nu}$ and $\hat{P}_{\mu}$ refers to Lorentz and translation generators respectively.

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[\hat{P}_{\mu}, \hat{P}_{\nu}] &= 0 \qquad (2)
\end{aligned}
$$

where $\hat{M}_{\mu\nu}$ and $\hat{P}_{\mu}$ refers to Lorentz and translation generators respectively.

Strategy:

Demand (2) remains undeformed

↓

Deform infinitesimal ISO(1,3) transformation, ensuring the covariance of space-time algebra

↓

Generators and the corresponding actions on space-time coordinates are deformed.

Infinitesimal transformation: $\delta X = \epsilon[G, X]$, where G and ϵ are generator and parameter for a certain transformation.

Ansatz for deformed transformations:

$$[\hat{M}_{\mu\nu}, \hat{X}_\rho] = i(\eta_{\nu\rho}\hat{X}_\mu - \eta_{\mu\rho}\hat{X}_\nu) + i\psi_{\mu\nu\rho}(\hat{P}, \hat{M}; a) \quad (3)$$

$$[\hat{P}_\mu, \hat{X}_\nu] = -i\eta_{\mu\nu}\,\phi(\hat{P}; a) + i\chi_{\mu\nu}(\hat{P}, \hat{M}; a) \quad (4)$$

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The deformations contained in ψ, ϕ, χ has to be chosen wisely.

(i) Dimensional consistency
(ii) Order of deformation parameter 'a'
(iii) Proper commutative limit.

M. Dimitrijevic, F. Meyer, L. Moller, J. Wess, Eur.Phys.J.C 36 (2004) 117-126.
S. Meljanac, A. Samsarov, M. Stojic, K.S. Gupta, Eur.Phys.J.C 53 (2008) 295-309.
T.R. Govindarajan, Kumar S. Gupta, E. Harikumar, S. Meljanac, J.Phys.Conf.Ser. 306 (2011) 012019; Phys.Rev.D 77 (2008) 105010.
S. Meljanac, A. Samsarov, J. Trampetic, M. Wohlgenannt, JHEP 12 (2011) 010.

Now we employ two kinds of Jacobi identities:

(A)$[X,[G,G]]+$ cyclic comb. $=0$

(B) $[G,[X,X]]+$ cyclic comb.$=0$ $\Rightarrow$ Stability of the spacetime algebra (1) under transformation.

Results:

$$[\hat{M}_{\mu\nu},\hat{X}_\rho] = i(\eta_{\nu\rho}\hat{X}_\mu - \eta_{\nu\rho}\hat{X}_\mu) - i(a_\mu\hat{M}_{\nu\rho} - a_\nu\hat{M}_{\mu\rho}) \qquad (5)$$

$$[\hat{P}_\mu,\hat{X}_\nu] = -i\eta_{\mu\nu}\left[a^\alpha\hat{P}_\alpha + \sqrt{1+a^2\hat{P}^2}\right] + ia_\mu\hat{P}_\nu; \quad \phi(\hat{P}) = a^\mu\hat{P}_\mu + \sqrt{1+a^2\hat{P}^2} \qquad (6)$$

For commutative limit $a_\mu \to 0$, the above deformed commutators reproduce

$$[\hat{M}_{\mu\nu},\hat{X}_\rho] = i(\eta_{\nu\rho}\hat{X}_\mu - \eta_{\mu\rho}\hat{X}_\nu); \qquad [\hat{P}_\mu,\hat{X}_\nu] = -i\eta_{\mu\nu} \qquad (7)$$

So the deformed transformations compatible with the space-time algebra (1) are given by

Deformed translation: $\delta \hat{X}^{\mu} = i\epsilon^{\alpha}[\hat{P}_{\alpha}, \hat{X}^{\mu}] = \epsilon^{\mu}\phi(\hat{P}) - (\epsilon_{\nu}a^{\nu})\hat{P}^{\mu}$

Deformed L.T: $\delta \hat{X}^{\mu} = -\frac{i}{2}\omega^{\alpha\beta}[\hat{M}_{\alpha\beta}, \hat{X}^{\mu}] = \omega^{\alpha\mu}\hat{X}_{\alpha} - \omega^{\alpha\beta}a_{\alpha}\hat{M}_{\beta}{}^{\mu}$

The transformations are not vector-like, translation is momentum dependent.

Action of both translations depend on momentum of the state it acts on $\rightarrow$ Worldlines of two particles with different momenta are translated, by a different, momentum-dependant amounts, which means that the two worldlines may cross for a local observer but miss each other for a translated observer$\rightarrow$ Relative Locality[a] $\leftarrow$ Curved momentum space.

[a] G.Amelino-Camelia, L Freidel, J. Kowalski-Glikman, L. Smolin, PRD 84, 084010 (2011)

"Demotion" of commutators to Dirac brackets

$$[\hat{f}, \hat{g}] \longrightarrow \{f, g\}_{D.B} = \lim_{\hbar \to 0} \frac{1}{i\hbar}[\hat{f}, \hat{g}] \tag{8}$$

$$\begin{aligned}
\{X^\mu, X^\nu\}_{D.B} &= \mathfrak{a}^\mu X^\nu - \mathfrak{a}^\nu X^\mu; \\
\{P_\mu, X^\nu\}_{D.B} &= -\delta_\mu{}^\nu \left[\mathfrak{a}^\alpha P_\alpha + \sqrt{1 + \mathfrak{a}^2 P^2}\right] + \mathfrak{a}_\mu P^\nu; \\
\{P_\mu, P_\nu\}_{D.B} &= 0
\end{aligned} \tag{9}$$

$$a^\mu \sim l_P = \sqrt{G\hbar} \to 0 \quad \text{when } \hbar, G \to 0$$

$$\mathfrak{a}^\mu = \lim_{\hbar, G \to 0} \frac{a^\mu}{\hbar} \sim \sqrt{\frac{G}{\hbar}} = \left[\frac{1}{M}\right] =^? \frac{1}{m_p} \ (\text{in } c = 1 \text{ unit})$$

Dirac Brackets → Constraint Matrix → Constraints → First order Lagrangian First Order Lagrangian:

$$L_f = f_\mu(X,P)\dot{X}^\mu + g_\mu(X,P)\dot{P}^\mu - H(P,X) \tag{10}$$

H= Hamiltonian of the system

Canonical momenta conjugate to X_μ and P_μ

$$\Pi^X_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = f_\mu, \quad \Pi^P_\mu = \frac{\partial L}{\partial \dot{P}^\mu} = g_\mu$$

fulfilling,

$$\{X_\mu, \Pi^X_\nu\} = \eta_{\mu\nu} = \{P_\mu, \Pi^P_\nu\} \tag{11}$$

Structure of Constraints:

$$\Sigma^1_\mu = \Pi^X_\mu - f_\mu(X,P) \approx 0; \qquad \Sigma^2_\mu = \Pi^P_\mu - g_\mu(X,P) \approx 0. \tag{12}$$

Let, $\xi_\mu^{(1)} = X_\mu$ and $\xi_\mu^{(2)} = P_\mu$

$$\{\xi_\mu^{(a)}, \xi_\nu^{(b)}\}_{DB} = \{\xi_\mu^{(a)}, \xi_\nu^{(b)}\} - \{\xi_\mu^{(a)}, \Sigma_\alpha^{(c)}\}(\Lambda^{-1})^{\alpha\beta}{}_{cd}\{\Sigma_\beta^{(d)}, \xi_\nu^{(b)}\} \qquad (13)$$

where $a, b = 1, 2; \quad \mu, \nu = 0, 1, 2, 3.$

Example

$$\theta_{\mu\nu} =: \mathfrak{a}_\mu X_\nu - \mathfrak{a}_\nu X_\mu = \{X_\mu, X_\nu\}_{DB} = 0 - \{X_\mu, \Sigma_\alpha^1\}(\Lambda^{-1})_{11}^{\alpha\beta}\{\Sigma_\beta^2, X_\nu\}$$

$$(\Lambda^{-1})^{\mu\nu}{}_{ab} = \begin{pmatrix} \theta^{\mu\nu} & \eta^{\mu\nu}\phi(P) - \mathfrak{a}^\nu P^\mu \\ \eta^{\mu\nu}\phi(P) + \mathfrak{a}^\mu P^\nu & 0 \end{pmatrix} \qquad (14)$$

$$\Lambda_{\mu\nu}{}^{ab} = \phi^{-1}(P) \begin{pmatrix} 0 & -\eta_{\mu\nu} - t(P)\mathfrak{a}_\mu P_\nu \\ \eta_{\mu\nu}(P) + t(P)\mathfrak{a}_\nu P_\mu & C_{\mu\nu} \end{pmatrix} \qquad (15)$$

where $t(P) = \frac{1}{\phi(P) - \mathfrak{a}.P}$

and $C_{\mu\nu} = \phi^{-1}(P)\left[\theta_{\mu\nu} + t(P)(\theta_{\mu\alpha}\mathfrak{a}^\alpha P_\nu - \theta_{\nu\alpha}\mathfrak{a}^\alpha P_\mu)\right]$

Constraint Matrix:

$$(\Lambda_{\mu\nu})^{ab} = \{\Sigma_{\mu}^{(a)}, \Sigma_{\nu}^{(b)}\} \tag{16}$$

$$f_{\mu} = 0; \qquad g_{\mu} = -\phi^{-1}(P)\left[X_{\mu} + \frac{(\mathfrak{a}.X)P_{\mu}}{\phi(P) - \mathfrak{a}.P}\right] \tag{17}$$

First order Lagrangian for relativistic free particle

$$L_f^{\tau} = -\phi^{-1}(P)\left[X^{\mu} + \frac{(\mathfrak{a}.X)P^{\mu}}{\phi(P) - \mathfrak{a}.P}\right]\dot{P}_{\mu} - e(\tau)(f(P^2) - M^2) \tag{18}$$

τ is the evolution parameter of the system and $e(\tau)$ is a Lagrangian multiplier enforcing the mass-shell condition $f(P^2) - M^2 = 0$ ($M =$ mass of the particle in κ space-time).

*$P^2 = m^2$ is the eigen-value of the Casimir of ISO(1,3) group.

$$L_f^\tau = -X^\beta\left[\phi^{-1}(P)\left(\delta_\beta{}^\mu + \frac{\mathfrak{a}_\beta P^\mu}{\phi(P)-\mathfrak{a}.P}\right)\right]\dot{P}_\mu - e(\tau)(f(P^2)-M^2)$$

$$L_f^\tau\, d\tau = -X^b E(P)_b{}^\mu dP_\mu - e(\tau)(f(P^2)-M^2)d\tau \qquad (19)$$

$$L_f^\tau\, d\tau = -X^\alpha \delta_\alpha{}^\beta dp_\beta - e(\tau)(P^2-m^2)d\tau \qquad \text{for usual free relativistic particle}$$

$e_b = E_b{}^\mu dP_\mu \;\Rightarrow\; de_b \neq 0 \;\rightarrow$ Non-holonomic basis in momentum space.

Presence of non-trivial $E_b{}^\mu(P)$ indicates that they may correspond non-trivial tetrads of "curved" momentum space.
It basically stems from

$$\{X^b, P_\mu\} = (E^{-1}(P))^b{}_\mu = \delta^b{}_\mu\,\phi(P) - \mathfrak{a}_\mu P^b$$
$$\{X^b, X^c\} = \mathfrak{a}^b X^c - \mathfrak{a}^c X^b$$

$$q^\beta = X^b E_b{}^\beta(P) \tag{20}$$

$$\{q^\mu, P_\nu\}_{D.B} = \delta^\mu{}_\nu, \qquad \{q^\mu, q^\nu\}_{D.B} = \{P_\mu, P_\nu\}_{D.B} = 0 \tag{21}$$

So we can construct a metric out of the tetrads in momentum space as

$$\tilde{g}_{\mu\nu}(P) = \eta_{ab}(E^{-1}(P))^a{}_\mu(E^{-1}(P))^b{}_\nu \tag{22}$$

$$\tilde{g}_{\mu\nu}(P) = \phi^2\eta_{\mu\nu} - \phi(\mathfrak{a}_\mu P_\nu + \mathfrak{a}_\nu P_\mu) + \mathfrak{a}^2 P_\mu P_\nu \tag{23}$$

Problem: Not covariant even under Lorentz transformation! -particularly because of the presence of a_μ's, which are not vectors.

$\rightarrow$ So the momentum space $\mathcal{P} \not\approx$ differentiable manifold.

$\rightarrow$ However, square of the geodesic distance between two points in momentum space $\mathcal{P} = D^2 = M^2$.

Geodesic distance:

$$D(0,P) = \int_0^P \sqrt{g_{\mu\nu}(p)dp^\mu dp^\nu} = \int_0^\tau d\tau' \sqrt{g_{\mu\nu}(p)\dot{p}^\mu \dot{p}^\nu} \ ; \dot{p}^\mu = \frac{dp^\mu}{d\tau'} \quad (24)$$

$\rightarrow$ The geodesic distance (24) in a curved manifold should be invariant under diffeomorphism, which cannot be achieved with the non-tensorial metric $\tilde{g}_{\mu\nu}(\tilde{P})$.

$\rightarrow$ Too extract any sensible meaning about the extremal distance, we can formally think of a covariantly transforming metric $g_{\mu\nu}(P)$ such that $\tilde{g}_{\mu\nu}(\tilde{P})$ will be treated as a particular form of the metric $g_{\mu\nu}(P)$ in a fiducial frame.

$\rightarrow$ Formally promote both $\mathfrak{a}^{\mu}$'s and P^{μ}'s to vectors under diffeomorphism, which will induce the following transformation in the metric tensor

$$\tilde{g}_{\alpha\beta}(\tilde{P}) \rightarrow g_{\mu\nu}(P) = \frac{\partial P^{\alpha}}{\partial P'^{\mu}} \frac{\partial P^{\beta}}{\partial P'^{\nu}} \tilde{g}_{\alpha\beta}(P) \tag{25}$$

$$\eta_{\alpha\beta} \rightarrow G_{\mu\nu} = \frac{\partial \tilde{P}^{\alpha}}{\partial P^{\mu}} \frac{\partial \tilde{P}^{\beta}}{\partial P^{\nu}} \eta_{\alpha\beta}$$

$$\tilde{P}^2 = \eta_{\alpha\beta} \tilde{P}^{\alpha} \tilde{P}^{\beta} = \mu^2 \rightarrow G_{\mu\nu} \frac{\partial P^{\mu}}{\partial \tilde{P}^{\alpha}} \frac{\partial P^{\nu}}{\partial \tilde{P}^{\beta}} \tilde{P}^{\alpha} \tilde{P}^{\beta} = \eta_{\alpha\beta} \tilde{P}^{\alpha} \tilde{P}^{\beta} = \mu^2 \tag{26}$$

With this

$$\tilde{g}_{\alpha\beta}(\tilde{P}) \tilde{P}^{\alpha} \tilde{P}^{\beta} = \tilde{P}^2 (1 + \mathfrak{a}^2 \tilde{P}^2) = \mu^2 (1 + \mathfrak{a}^2 \mu^2) \tag{27}$$

becomes invariant.

The mass-shell condition is defined as $C = D^2 = M^2$.

Distance function $D(P) := D(0, P)$ satisfies following differential equation

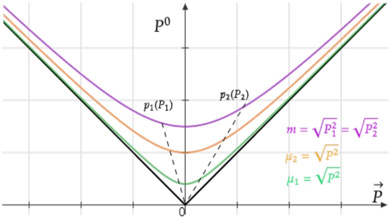
$$\partial^\mu D(P)\, g_{\mu\nu}(P)\, \partial^\nu D(P) = 1$$

Equivalently,

$$\partial^\mu C(P)\, g_{\mu\nu}(P)\, \partial^\nu C(P) = 4C \tag{28}$$

To solve it we make the following ansatz: $C = f(P^2)$

Geometrically this just means that $D(0, P_1) = D(0, P_2)$, if P_1 and P_2 both belong to the same hyperboloid: $P_1^2 = P_2^2 = m^2$



$$M = \sqrt{C} = D(0,P) = \frac{1}{2}\int_0^{P^2=m^2} \frac{d(\mu^2)}{\sqrt{\mu^2(1+\mathfrak{a}^2\mu^2)}} = \int_0^m \frac{d\mu}{\sqrt{1+\mathfrak{a}^2\mu^2}}, \qquad (29)$$

$\mathfrak{a}^2 = \eta_{\mu\nu}\mathfrak{a}^\mu\mathfrak{a}^\nu$

Case-1 ($\mathfrak{a}^2 = 0$)

It follows quite trivially from (29), there is no noncommutative effect in the dispersion relation as $M = m = \sqrt{P^2}$.

Case-2 ($\mathfrak{a}^2 < 0$)

In this case (29) can be simplified to

$$M = \frac{1}{\sqrt{-\mathfrak{a}^2}}\left[\sin^{-1}(m\sqrt{-\mathfrak{a}^2})\right] \qquad (30)$$

Taylor series expansion around the commutative limit $\mathfrak{a} \to 0$, is given by

$$M = \frac{1}{\sqrt{-\mathfrak{a}^2}}\left[\lambda + \frac{\lambda^2}{6} + \frac{3\lambda^4}{40} + \ldots\right], \qquad \text{for } \lambda = m\sqrt{-\mathfrak{a}^2} < 1 \qquad (31)$$

Since $\sin^{-1}\lambda$ for $\lambda > 1$ is undefined, $m < \frac{1}{\sqrt{-\mathfrak{a}^2}}$.

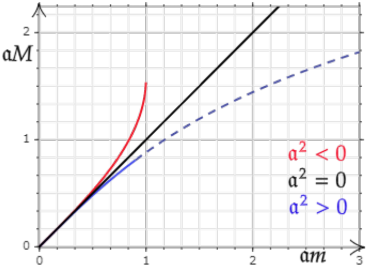
The corresponding bound for M is given by $M < \frac{\pi}{2\sqrt{-\mathfrak{a}^2}}$.

Case-3 ($\mathfrak{a}^2 > 0$)

$$M = \frac{1}{\mathfrak{a}} \sinh^{-1}(\mathfrak{a}m); \qquad \mathfrak{a} = \sqrt{\mathfrak{a}^2} \tag{32}$$

$$\sinh^{-1}\xi = \begin{cases} \xi - \frac{\xi^3}{6} + \frac{3\xi^5}{40} - \vartheta(\xi^7) + \ldots & \text{for } |\xi| < 1 \\ \pm\left[ln|2\xi| + \frac{1}{4\xi^2} - \frac{3}{32\xi^4} + \vartheta(\xi^{-6}) - \ldots \right] & \text{for } \pm\xi \geq 1 \end{cases} \tag{33}$$

where $\xi := \mathfrak{a}m$.



Relativistic and non-relativistic limits of κ Minkowski spacetime algebra

$$[\widehat{X}^{\mu}, \widehat{P}_{\nu}] = iE^{-1}\left(\widehat{P}; a\right)^{\mu}_{\nu}$$

Map between commutative and non-commutative coordinates:

$$X^{\mu} = \left(E^{-1}(P; a)\right)^{\mu}{}_{\alpha}\, q^{\alpha}$$

Spacetime contraction in $c \to \infty$ limit

$$q^0 \longrightarrow q^0_g = \lim_{c\to\infty} t(c) = \lim_{c\to\infty} \left(\frac{q^0}{c}\right); \qquad q^i \longrightarrow q^i_g = q^i$$

Inverse metric:

$$\gamma^{-1} := \lim_{c\to\infty} \eta^{-1}(q^0, q^i; c) = \lim_{c\to\infty} \left(\frac{1}{c^2}\frac{\partial}{\partial t} \otimes \frac{\partial}{\partial t} - \delta^{ij}\frac{\partial}{\partial q^i} \otimes \frac{\partial}{\partial q^j}\right)$$

$$= -\delta^{ij}\frac{\partial}{\partial q^i_g} \otimes \frac{\partial}{\partial q^j_g}$$

Spacetime contraction in $c \to 0$ limit:

$$q^0 \longrightarrow q^0_c = \lim_{c\to 0} t(c) = \lim_{c\to 0} \left(\frac{q^0}{c}\right); \qquad q^i \longrightarrow q^i_c = q^i$$

Metric:

$$g = \lim_{c\to 0} \eta(q^0, q^i; c) = \lim_{c\to 0} \left(c^2 dt \otimes dt - \delta_{ij} dq^i \otimes dq^j\right) = -\delta_{ij} dq^i_c \otimes dq^j_c$$

κ-Galilean spacetime

$$\hat{X}^0 \longrightarrow \hat{X}^0_g = \lim_{c\to\infty} \hat{T}(c) = \lim_{c\to\infty} \left(\frac{\hat{X}^0}{c}\right); \qquad \hat{X}^i \longrightarrow \hat{X}^i_g = \hat{X}^i$$

$$\left[\frac{\hat{X}^0}{c}, \hat{X}^i\right] = i\left(\frac{a^0}{c}\hat{X}^i - a^i\frac{\hat{X}^0}{c}\right) \tag{34}$$

$$\left[\hat{X}^0_g, \hat{X}^i_g\right] = -ia^i\hat{X}^0_g \tag{35}$$

$$\left[\hat{X}^i_g, \hat{X}^j_g\right] = i\left(a^i\hat{X}^j_g - a^j\hat{X}^i_g\right) \tag{36}$$

Non-relativistic limit of κ Minkowski Bopp map:

$$\hat{X}^0_g = q^0_g\left(-\vec{a}\cdot\widehat{\vec{P}}_g + \sqrt{1+(\vec{a})^2\left(\widehat{\vec{P}}_g\right)^2}\right)$$

$$\hat{X}^0_g = q^0_g\left(-\vec{a}\cdot\widehat{\vec{P}}_g + \sqrt{1+(\vec{a})^2\left(\widehat{\vec{P}}_g\right)^2}\right)$$

For consistency we must set $a^0 = 0, a^i \neq 0 \Rightarrow$ space-like NC parameter

Coordinate algebra:

$$\left[\widehat{X}_g^0, \widehat{X}_g^i\right] = -ia^i \widehat{X}_g^0; \quad \left[\widehat{X}_g^i, \widehat{X}_g^j\right] = i\left(a^i \widehat{X}_g^j - a^j \widehat{X}_g^i\right)$$

Phase space Algebra

$$\left[\widehat{P}_0, \widehat{X}^0\right] = -i\phi\left(\vec{a}, \widehat{\vec{P}}\right); \quad \phi\left(\vec{a}, \widehat{\vec{P}}\right) = -\vec{a}\cdot\widehat{\vec{P}} + \sqrt{1 + (\vec{a})^2\left(\widehat{\vec{P}}\right)^2}$$

$$\left[\widehat{P}_0, \widehat{X}^k\right] = 0; \quad \left[\widehat{P}_i, \widehat{X}^0\right] = 0$$

$$\left[\widehat{P}_i, \widehat{X}^k\right] = -i\delta_i^k \phi\left(\vec{a}, \widehat{\vec{P}}\right) + ia_i \widehat{P}^k$$

$$\widehat{X}^0 \longrightarrow \widehat{X}^0_c = \lim_{c\to 0} \widehat{T}(c) = \lim_{c\to 0} \left(\frac{\widehat{X}^0}{c}\right); \qquad \widehat{X}^i \longrightarrow \widehat{X}^i_c = \widehat{X}^i$$

$$\lim_{c\to 0} \left[\frac{\widehat{X}^0}{c}, \widehat{X}^j\right] = i \lim_{c\to 0} \left(\left(\frac{a^0}{c}\right)\widehat{X}^j - a^j \frac{\widehat{X}^0}{c}\right)$$

$$a^0 \longrightarrow a^0_c = \lim_{c\to 0} \left(\frac{a^0}{c}\right) \sim t_p = \sqrt{\frac{\hbar G}{c^5}} \sim 10^{-44}\ s; \qquad a^i \longrightarrow a^i_c = a^i \tag{37}$$

$$\left[\widehat{X}^0_c, \widehat{X}^j_c\right] = i\left(a^0_c \widehat{X}^j_c - a^j_c \widehat{X}^0_c\right)$$
$$\left[\widehat{X}^i_c, \widehat{X}^j_c\right] = i\left(a^i_c \widehat{X}^j_c - a^j_c \widehat{X}^i_c\right)$$

$$\widehat{X}_c^0 = q_c^0\sqrt{1+(a_c^0)^2\left(\widehat{P}_0^c\right)^2}$$

$$\widehat{X}_c^i = q_c^i\left(a_c^0\widehat{P}_0^c + \sqrt{1+(a_c^0)^2\left(\widehat{P}_0^c\right)^2}\right)$$

For consistency we must set $a^i = 0,\ a_c^0 \neq 0 \Rightarrow$ time-like NC parameter.

Coordinate algebra:

$$\left[\widehat{X}_c^0, \widehat{X}_c^i\right] = i a_c^0 \widehat{X}_c^i; \qquad \left[\widehat{X}_c^i, \widehat{X}_c^j\right] = 0$$

Phase space Algebra

$$\left[\widehat{P}_0, \widehat{X}^0\right] = -i\sqrt{1+a^2\left(\widehat{P}_0\right)^2}; \left[\widehat{P}_i, \widehat{X}^k\right] = -i\delta_i^k\left(a\widehat{P}_0 + \sqrt{1+a^2\left(\widehat{P}_0\right)^2}\right)$$

T. Trzesniewski, JHEP 2024 (2024) 200.

Summary

Deformed Poincare symmetry compatible with κ deformed space-time algebra

$\downarrow$

Momentum dependent transformation $\rightarrow$ Indication of Relative locality and curved momentum space

$\downarrow$

Free particle Lagrangian construction to identify the momentum space viel-bein

$\downarrow$

Momentum space metric and geodesic distance

$\downarrow$

Deformed dispersion relation

Summary

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- It is quite interesting to see how one can construct multi-particle actions in presence of interactions (taken to be simple collisions) and explore more features of momentum space geometry.

P Nandi, AC, S K Pal, B Chakraborty, F G Scholtz, Symmetries of κ Minkowski space-time: A possibility of exotic momentum space geometry?, JHEP 07 (2023) 142.

Deeponjit Bose, **AC**, Biswajit Chakraborty, *arXiv:2401.05769*

THANK YOU