Black Holes and Noncommutativity

Noncommutative gravity, spacetime perturbations and NC gauge theory
(arXiv:2409.01402 and one more to appear)

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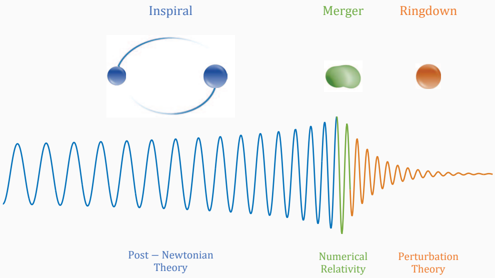
Black hole perturbations

Understanding Quantum Effects in Black Holes

Croatian Science Foundation research project
Search for Quantum Spacetime in Black Hole QNM Spectrum and Gamma Ray Bursts

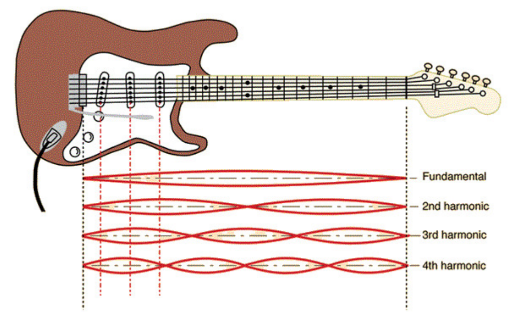
The aim of this project is to investigate the QNMs resulting from perturbations of realistic 4-dimensional black holes in the presence of quantized spacetime.



Compact Binary Coalescence, relevant to quantum effects in spacetime (arXiv:1610.03567)

Black Hole Perturbations and Quasinormal Modes (QNM)

- QNM frequency: $\omega = \omega_R + i\omega_I$. ω_R: oscillation frequency; ω_I: damping rate.
- QNMs depend only on black hole parameters ("footprints" of a black hole).
- Schwarzschild black holes (Regge & Wheeler):
  linearised Einstein equations $\rightarrow$ Schrödinger-like equation.



Vibrating string analogy

Wave Equation for Perturbations

$$\frac{\partial^2 y}{\partial t^2} - v^2 \frac{\partial^2 y}{\partial x^2} = 0, \quad y(t,0) = y(t,L) = 0, \quad f_n = \frac{nv}{2L}$$

Discrete frequencies depend on system length (L) and wave speed (v).

Black Hole Perturbations: Approach and Boundary Conditions

Approach to BH Perturbations:

- Compute equations of motion for perturbations.
- Cast into a wave propagation equation.
- Derive boundary conditions.
- Perform numerical computation.

Boundary Conditions:

- At event horizon ($r = r_H$): impose ingoing conditions: $h \sim e^{-i(\omega t + kr)}$.
- At infinity ($r = \infty$): impose outgoing conditions: $h \sim e^{-i(\omega t - kr)}$.

Key Differences:

Guitar String :: BH Perturbations

Self-adjoint :: Not self-adjoint

Real B.C. :: Complex B.C.

Outcome:

BH perturbations yield damped/exponentially growing sinusoids, indicating energy loss toward the horizon and infinity.

Metric Perturbations and Axial Modes

- Black hole metric: $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu}$ being the perturbation.
- Schwarzschild background: $ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$
- $h_{\mu\nu}$ decomposed into spherical harmonics $Y_{\ell m}$; axial (odd-parity) and polar (even-parity) modes are treated separately.
- Time dependence handled via Fourier modes: $F(t,r) = \int d\omega \tilde{F}(\omega, r) e^{-i\omega t}$.

Gauge and Axial Modes:

- Gauge freedom arises from diffeomorphism invariance: $h_{\mu\nu} \to h_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$.
- Regge-Wheeler gauge used for axial modes.
- Axial perturbations for $\ell \geq 2$ parameterized by $h_0^{\ell m}$, $h_1^{\ell m}$, and $h_2^{\ell m}$.

Key Equations:

- $h_{t\theta} \propto h_0^{\ell m} \partial_\varphi Y_{\ell m}$
- $h_{r\theta} \propto h_1^{\ell m} \partial_\varphi Y_{\ell m}$
- $h_{t\varphi} \propto h_0^{\ell m} \partial_\theta Y_{\ell m}$
- $h_{r\varphi} \propto h_1^{\ell m} \partial_\theta Y_{\ell m}$

Schrödinger-like Equation and Effective Potential

Perturbation Equation:

$$\frac{dY}{dr} = M(r)Y, \quad M(r) = \begin{pmatrix} \frac{2}{r} & 2i\lambda\frac{r-R}{r^3} - i\omega^2 \\ -i\frac{r^2}{(r-R)^2} & -\frac{R}{r(r-R)} \end{pmatrix}$$

where $Y = {}^T(h_0(r), h_1(r)/\omega)$.

Schrodinger-like equation:

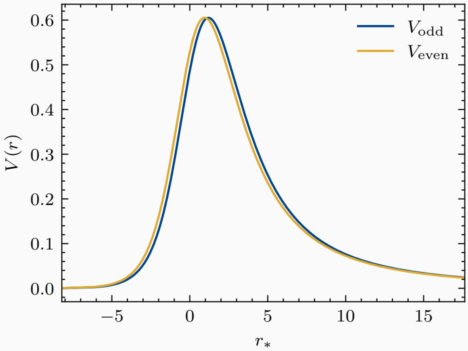
$$\frac{d^2\hat{Y}_1}{dr_*^2} + (\omega^2 - V(r))\hat{Y}_1 = 0$$

Tortoise Coordinate:

$$r_* = r + R\ln\left(\frac{r}{R} - 1\right)$$

Effective Potential:

$$V_{\text{odd}}(r) = \left(1 - \frac{R}{r}\right)\frac{2(\lambda+1)r - 3R}{r^3}$$



Effective potential for axial perturbations.

Noncommutative differential geometry

Lie Algebra, Hopf Algebra, and Drinfeld Twist

Lie Algebra and Hopf Algebra:

- Lie algebra of vector fields $(\Xi, [\cdot, \cdot])$: describes infinitesimal diffeomorphisms of $\mathcal{M}$.
- Universal enveloping algebra $U\Xi$ encodes Leibniz rule, inverse, and normalization via Δ (coproduct), S (antipode), ϵ (counit).
- Hopf algebra structure: $H = (U\Xi, \mu, \Delta, \epsilon, S)$.
- Hopf algebra conditions:

$$(\Delta \otimes \mathrm{id})\Delta(\xi) = (\mathrm{id} \otimes \Delta)\Delta(\xi)$$
$$(\epsilon \otimes \mathrm{id})\Delta(\xi) = \xi = (\mathrm{id} \otimes \epsilon)\Delta(\xi)$$
$$\mu((S \otimes \mathrm{id})\Delta(\xi)) = \epsilon(\xi)1$$

Drinfeld Twist and R-Matrix:

- Drinfeld twist $\mathcal{F} \in H \otimes H$: deforms the Hopf algebra.
- Deformed Hopf algebra:

$$\Delta^{\mathcal{F}}(\xi) = \mathcal{F}\Delta(\xi)\mathcal{F}^{-1}, \quad S^{\mathcal{F}}(\xi) = \chi S(\xi)\chi^{-1}$$

- Universal R-matrix relates the deformed coproduct to its coopposite:

$$(\Delta^{\mathcal{F}})^{\mathrm{cop}}(\xi) = R\Delta^{\mathcal{F}}(\xi)R^{-1}$$

where $R = \mathcal{F}_{21}\mathcal{F}^{-1}$.

Noncommutative Space, Moyal-Weyl Twist, and $\star$-Product

Noncommutative Space:

- The deformed Hopf algebra $H^{\mathcal{F}}$ is not cocommutative, leading to NC structures on spaces, like the algebra of functions on $\mathcal{M}$.

Moyal-Weyl Twist:

- We use the Moyal-Weyl twist $\mathcal{F}$ on $\mathcal{M} = \mathbb{R}^N$.
- The twist is given by:

$$\mathcal{F} = \exp\left(-\frac{ia}{2}\Theta^{\mu\nu}\partial_\mu \otimes \partial_\nu\right)$$

where $\Theta^{\mu\nu}$ is antisymmetric.

$\star$-Product:

- The algebra of smooth functions $\mathcal{C}^\infty(\mathcal{M})$ with pointwise multiplication $h(x)k(x)$ becomes $\mathcal{A}^\star = (\mathcal{C}^\infty(\mathcal{M}), \star)$.
- The $\star$-product for the Moyal-Weyl twist is:

$$h \star k = h\, e^{\frac{ia}{2}\overleftarrow{\partial}_\mu \Theta^{\mu\nu}\overrightarrow{\partial}_\nu}\, k$$

- $\mathcal{A}^\star$ is R-symmetric:

$$h \star k = \bar{R}^\alpha(k) \star \bar{R}_\alpha(h)$$

NC Geometry: Covariant Derivatives, Torsion, and Curvature

- $\star$-**Covariant Derivative:** A $\star$-covariant derivative $\nabla^\star$ along $v \in \Xi$ is a $\mathbb{C}$-linear map satisfying:

$$\nabla^\star_{v+w} z = \nabla^\star_v z + \nabla^\star_w z,$$
$$\nabla^\star_{h\star v} z = h \star \nabla^\star_v z,$$
$$\nabla^\star_v (h \star z) = \pounds^\star_v(h) \star z + \bar{R}^\alpha(h) \star \nabla^\star_{\bar{R}_\alpha(v)} z$$

- $\star$-**Torsion and Curvature:** Given $\nabla^\star$, the $\star$-torsion $T^\star$ and $\star$-curvature $R^\star$ are:

$$T^\star(v,w) = \nabla^\star_v w - \nabla^\star_{\bar{R}^\alpha(w)} \bar{R}_\alpha(v) - [v,w]_\star,$$
$$R^\star(v,w,z) = \nabla^\star_v \nabla^\star_w z - \nabla^\star_{\bar{R}^\alpha(w)} \nabla^\star_{\bar{R}_\alpha(v)} z - \nabla^\star_{[v,w]_\star} z$$

- $\star$-**Ricci Tensor:** In the $\star$-dual basis $\langle \partial_\mu, dx^\nu \rangle_\star = \delta^\nu_\mu$, the $\star$-Ricci tensor is:

$$R^\star(v,w) = \langle dx^\mu, R^\star(\partial_\mu, v, w) \rangle_\star$$

The noncommutative Ricci tensor is not R-symmetric, as the Riemann tensor is not R-antisymmetric in its last two indices.

NC Geometry: Moyal-Weyl Twist, Torsion, Curvature, and Inverse Metric

- **Moyal-Weyl Twist:** The $\star$-covariant derivative, torsion, curvature, and Ricci tensor are given by:

$$\nabla^{\star}_{\partial_\mu}\partial_\nu = \Gamma^{\star\rho}_{\mu\nu} \star \partial_\rho = \Gamma^{\star\rho}_{\mu\nu}\partial_\rho$$

$$T^{\star}(\partial_\mu, \partial_\nu) = \left(\Gamma^{\star\rho}_{\mu\nu} - \Gamma^{\star\rho}_{\nu\mu}\right)\partial_\rho,$$

$$R^{\star}(\partial_\mu, \partial_\nu, \partial_\rho) = \left(\partial_\mu \Gamma^{\star\sigma}_{\nu\rho} - \partial_\nu \Gamma^{\star\sigma}_{\mu\rho} + \Gamma^{\star\tau}_{\nu\rho} \star \Gamma^{\star\sigma}_{\mu\tau} - \Gamma^{\star\tau}_{\mu\rho} \star \Gamma^{\star\sigma}_{\nu\tau}\right)\partial_\sigma,$$

$$R^{\star}(\partial_\nu, \partial_\rho) = \partial_\mu \Gamma^{\star\mu}_{\nu\rho} - \partial_\nu \Gamma^{\star\mu}_{\mu\rho} + \Gamma^{\star\tau}_{\nu\rho} \star \Gamma^{\star\mu}_{\mu\tau} - \Gamma^{\star\tau}_{\mu\rho} \star \Gamma^{\star\mu}_{\nu\tau}.$$

- **Inverse Metric:** The metric and inverse metric satisfy:

$$g_{\mu\nu} \star g^{\nu\rho} = \delta^\rho_\mu, \quad g^{\mu\nu} \star g_{\nu\rho} = \delta^\mu_\rho$$

The inverse metric is:

$$g^{\star\alpha\beta} = g^{\alpha\beta} - g^{\gamma\beta}\Theta^{\mu\nu}(\partial_\mu g^{\alpha\sigma})(\partial_\nu g_{\sigma\gamma}) + \mathcal{O}(a^2)$$

- **Levi-Civita Connection:** The Levi-Civita connection is:

$$\Gamma^{\star\rho}_{\mu\nu} = \frac{1}{2} g^{\star\rho\sigma} \star \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)$$

Noncommutative gravitational perturbations

Non-commutative Einstein Manifolds and R-Symmetrization

- In commutative gravity, black hole perturbations use $R_{\mu\nu} = 0$.
- In non-commutative gravity, the analogous condition is $\mathcal{R}_{\mu\nu} = 0$.

- For Moyal-Weyl type deformations, R-symmetrization is:

$$\mathcal{R}_{\mu\nu} = R^\star_{(\mu\nu)} = \frac{1}{2}(R^\star_{\mu\nu} + R^\star_{\nu\mu})$$

R-symmetrized Ricci tensor

$$\mathcal{R}_{\mu\nu} \equiv \frac{1}{2}\left\langle dx^\alpha, R^\star(\partial_\alpha, \partial_\mu, \partial_\nu) + R^\star(\partial_\alpha, \bar{R}^A(\partial_\nu), \bar{R}_A(\partial_\mu))\right\rangle_\star$$

- Ensuring $R^\star_{\mu\nu} = 0$ is non-trivial due to the non-symmetry of $R^\star_{\mu\nu}$.

- A generalized abelian twist is:

$$\mathcal{F} = \exp\left(-\frac{ia}{2}\Theta^{\mu\nu} X_\mu \otimes X_\nu\right)$$

Selection of a Specific Twist and Semi-pseudo-Killing Twist

Selection of a Specific Twist:

- Twist selection in quantum gravity should ideally be guided by experiments, but in their absence, symmetry arguments provide guidance.
- A twist constructed from Killing vectors K^μ of the background $g_{\mu\nu}$ does not produce nontrivial NC effects since $\pounds_K g = 0$. The same applies to semi-Killing twists with K^μ and arbitrary vector V^ν.

Semi-pseudo-Killing Twist:

- In black hole perturbation studies $(g_{\mu\nu} + h_{\mu\nu})$, a Killing twist results in NC corrections that are quadratic in h, as $\pounds_K g = 0$ but $\pounds_K h \neq 0$.
- A semi-pseudo-Killing twist, constructed from Killing vector K^μ and arbitrary vector X^ν, yields leading linearized NC perturbation terms.

Semi-pseudo-Killing Twist

Twist Form and Killing Fields:

- The semi-pseudo-Killing twist is defined as:

$$\mathcal{F} = e^{-i\frac{a}{2}\left(K\otimes X - X\otimes K\right)}$$

- The background has two Killing fields K_t and K_φ. We choose:

$$K = \alpha\partial_t + \beta\partial_\varphi, \quad X = \partial_r$$

- The resulting commutation relations:

$$[t \stackrel{\star}{,} r] = ia\alpha, \quad [\varphi \stackrel{\star}{,} r] = ia\beta$$

Eigenvalue and $\star$-Product:

- The parameter λ is the eigenvalue of the Killing field's action on the perturbation:

$$h_{\mu\nu} \propto e^{-i\omega t} e^{im\varphi}, \quad \pounds_K h_{\mu\nu} = i\lambda h_{\mu\nu},$$

$$\lambda = -\alpha\omega + \beta m$$

- The linearized $\star$-product is:

$$h \star k = hk + \frac{i}{2}a\left[K(h)X(k) - X(h)K(k)\right]$$

- Where $X(k) \equiv \pounds(k)$.

Zerilli Gauge and Linearized Einstein Equations

- **Zerilli Gauge:** In polar perturbations, the metric is parameterized by four functions in the Zerilli gauge: $H_0^{\ell m}$, $H_1^{\ell m}$, $H_2^{\ell m}$, and $K^{\ell m}$:

$$\begin{aligned} h_{tt} &= A(r)\sum_{\ell,m} H_0^{\ell m}(t,r)Y_{\ell m}(\theta,\varphi), \quad h_{tr} = \sum_{\ell,m} H_1^{\ell m}(t,r)Y_{\ell m}(\theta,\varphi), \\ h_{rr} &= \frac{1}{A(r)}\sum_{\ell,m} H_2^{\ell m}(t,r)Y_{\ell m}(\theta,\varphi), \quad h_{ab} = \sum_{\ell,m} K^{\ell m}(t,r)g_{ab}Y_{\ell m}(\theta,\varphi), \end{aligned}$$

  where $A(r) = 1 - R/r$ and a, b denote angular coordinates.

- **Linearized Einstein Equations:** The equations of motion, up to linear order in perturbation $h_{\mu\nu}$ and NC deformation a, yield coupled PDEs for H_0, H_1, H_2, and K. Using $R = H_1/\omega$, the linearized Einstein equations reduce to:

$$\begin{aligned} K' &= \left[\alpha_0(r) + \alpha_2(r)\omega^2\right] K + \left[\beta_0(r) + \beta_2(r)\omega^2\right] R, \\ R' &= \left[\gamma_0(r) + \gamma_2(r)\omega^2\right] K + \left[\delta_0(r) + \delta_2(r)\omega^2\right] R \end{aligned}$$

  where $\alpha(r)$, $\beta(r)$, $\gamma(r)$, and $\delta(r)$ are complicated functions of r.

Linearized Einstein Equations and Schrödinger-Like Form

- We introduce the field redefinition:

$$K = \hat{f}(r)\hat{K} + \hat{g}(r)\hat{R}, \quad R = \hat{h}(r)\hat{K} + \hat{l}(r)\hat{R}$$

with the coordinate transformation $dr/d\hat{r}^* = \hat{n}(r)$.

- The requirement:

$$\frac{d\hat{K}}{d\hat{r}^*} = \hat{R}, \quad \frac{d\hat{R}}{d\hat{r}^*} = (V - \omega^2)\hat{K}$$

leads to the Schrödinger-like equation:

$$\frac{d^2\hat{K}}{d\hat{r}^{*2}} + (\omega^2 - V)\hat{K} = 0.$$

Coupled ODEs and Constraints:

- A generic transformation results in coupled ODEs:

$$\hat{K}' = \left[\hat{\alpha}_0(r) + \hat{\alpha}_2(r)\omega^2\right]\hat{K} + \left[\hat{\beta}_0(r) + \hat{\beta}_2(r)\omega^2\right]\hat{R},$$

$$\hat{R}' = \left[\hat{\gamma}_0(r) + \hat{\gamma}_2(r)\omega^2\right]\hat{K} + \left[\hat{\delta}_0(r) + \hat{\delta}_2(r)\omega^2\right]\hat{R}.$$

- Constraints:

$$\hat{\alpha}_0(r) = \hat{\alpha}_2(r) = \hat{\beta}_2(r) = \hat{\delta}_0(r) = \hat{\delta}_2(r) = 0,$$

$$\hat{\beta}_0(r) = 1, \quad \hat{\gamma}_2(r) = -1.$$

- Seven conditions for five unknowns $(\hat{f}(r), \hat{g}(r), \hat{h}(r), \hat{l}(r), \hat{n}(r))$ allow the system to admit a solution.

Non-commutative Solutions and Zerilli Potential

- The solutions to the required field redefinition are:

$$\hat{f}(r) = f(r) + \lambda a \tilde{f}(r), \quad \hat{g}(r) = g(r) + \lambda a \tilde{g}(r),$$
$$\hat{h}(r) = h(r) + \lambda a \tilde{h}(r), \quad \hat{l}(r) = l(r) + \lambda a \tilde{l}(r),$$

where the terms with tildes are the non-commutative corrections.

- The tortoise coordinate is:

$$\hat{n}(r) = \frac{r}{r-R} - \lambda a \frac{4r^2 + 2\Lambda r R + 3R^2}{2(r-R)^2(2\Lambda r + 3R)}$$

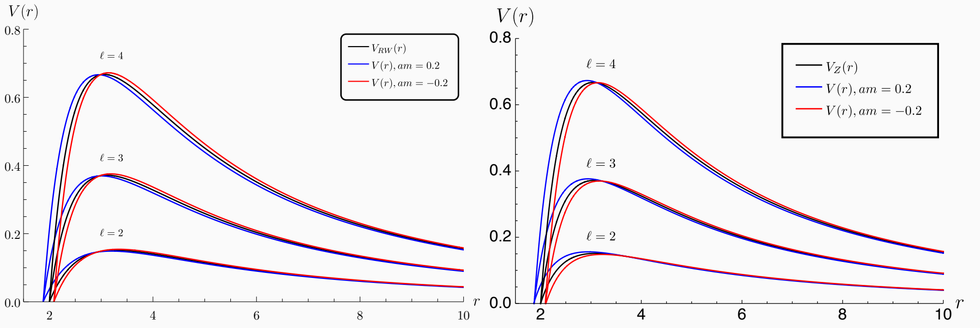
- The effective potential is $V = V_Z + V_{NC}$, where:

$$V_Z = \frac{(r-R)\left(8\Lambda^2(\Lambda+1)r^3 + 12\Lambda^2 r^2 R + 18\Lambda r R^2 + 9R^3\right)}{r^4(2\Lambda r + 3R)^2}$$

$$V_{NC} = \frac{\lambda a}{4r^5(2\Lambda r + 3R)^3}\left[-32\Lambda^2(2\Lambda^2 + 7)r^5 + \cdots + 387R^5\right]$$

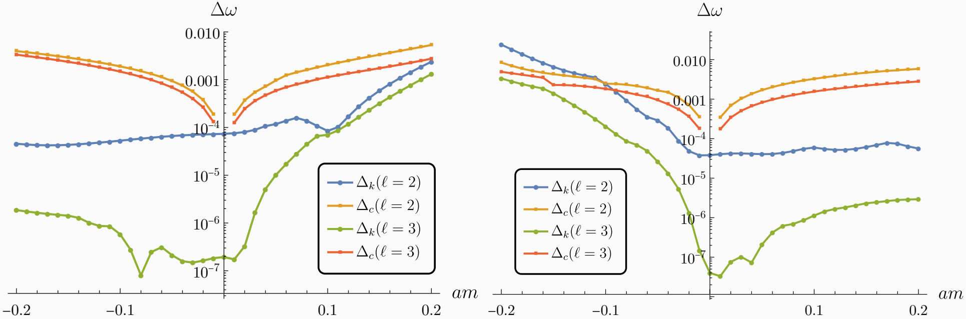
(NC potential abbreviated.)

Non-commutative potentials



Left: NC Regge-Wheeler potential. Right: NC Zerilli potential

QNM using higher order WKB method



The comparison between the noncommutative correction Δ_c and the relative error Δ_k in the optimal WKB order is illustrated. Left: Axial case. Right: Polar case.

The WKB QNM formula

$$\frac{i(\omega^2 - V_0)}{\sqrt{-2V_0''}} - \sum_{i=2}^{6} \tilde{\Lambda}_i = n + \frac{1}{2}$$

WKB error formula

$$\Delta_k = \frac{|\omega_{k+1} - \omega_{k-1}|}{2},$$

NC correction Δ_c

$$\Delta_c = |\omega_{NC} - \omega_C|.$$

Axial QNM frequencies

am	WKB	Order	Pöschl-Teller	Rosen-Morse
-0.2	0.3775(61) - 0.0883(97) i	6	0.382049 - 0.090320 i	0.38335 - 0.08924 i
-0.1	0.3755(14) - 0.0887(70) i	6	0.380114 - 0.090466 i	0.38057 - 0.09008 i
-0.01	0.3738(07) - 0.0888(92) i	6	0.378454 - 0.090521 i	0.37855 - 0.09044 i
-0.001	0.3736(38) - 0.0888(91) i	6	0.378294 - 0.090520 i	0.37838 - 0.09044 i
0	0.3736(19) - 0.0888(91) i	6	0.378276 - 0.090520 i	0.37837 - 0.09044 i
0.001	0.3736(01) - 0.0888(91) i	6	0.378258 - 0.090520 i	0.37838 - 0.09042 i
0.01	0.3734(33) - 0.0888(88) i	6	0.378099 - 0.090518 i	0.37836 - 0.09030 i
0.1	0.3715(87) - 0.0889(38) i	4	0.376562 - 0.090455 i	0.37756 - 0.08961 i
0.2	0.36(8345) - 0.08(8195) i	4	0.375007 - 0.090238 i	0.37611 - 0.08930 i

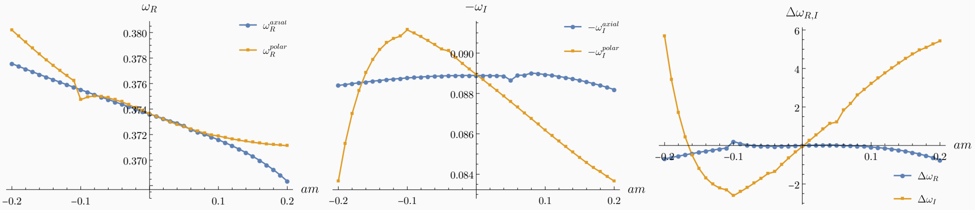
Table 1: NC axial QNMs for $n = 0$, $M = 1$ ($R = 2$), and $\ell = 2$. To convert frequencies to kHz, multiply by $2\pi \times 5142\,\text{Hz} \times (M_\odot/M)$. For a $10M_\odot$ black hole, $M\omega \approx (0.37, -0.09)$ corresponds to 1.2 kHz and a damping time of 0.55 ms. LIGO detects frequencies from 10 Hz to 10 kHz.

Polar QNM frequencies

am	WKB	Order	Pöschl-Teller	Rosen-Morse
-0.2	0.3(80198) - 0.0(83646) i	3	0.382642 - 0.097531 i	0.38379 - 0.09648 i
-0.1	0.37(4735) - 0.09(1148) i	4	0.380292 - 0.093609 i	0.38178 - 0.09230 i
-0.01	0.3738(64) - 0.0892(07) i	5	0.378475 - 0.090866 i	0.37890 - 0.09050 i
-0.001	0.3736(58) - 0.0889(67) i	5	0.378308 - 0.090622 i	0.37845 - 0.09050 i
0	0.3736(36) - 0.0889(40) i	5	0.378290 - 0.090595 i	0.37839 - 0.09051 i
0.001	0.3736(13) - 0.0889(14) i	5	0.378272 - 0.090567 i	0.37836 - 0.09049 i
0.01	0.3734(13) - 0.0886(75) i	5	0.378109 - 0.090322 i	0.37821 - 0.09023 i
0.1	0.3718(88) - 0.0861(75) i	6	0.376612 - 0.088102 i	0.37741 - 0.08744 i
0.2	0.3711(29) - 0.0836(58) i	7	0.375215 - 0.085959 i	0.37794 - 0.08379 i

Table 2: Table of NC polar QNMs for $n = 0$, $M = 1$, and $\ell = 2$.

Isospectral Breaking



(a) ω_R vs. am (b) ω_I vs. am (c) Relative deviation $\Delta\omega_{R,I}$

Isospectrality breaking due to noncommutativity for $\ell = 2$. Parameters: $n = 0$, $M = 1$ $(R = 2)$. Values chosen correspond to optimal WKB order.

Relative deviation, $\Delta\omega_{R,I} = 100 \times \dfrac{\omega_{R,I}^{axial} - \omega_{R,I}^{polar}}{\omega_{R,I}^{polar}}$.

What's Next?

- **Perturbations of Spinning Black Holes**
  - *Teukolsky Approach*: Using the Newman–Penrose formalism
  - *Slow Rotating Approximation*
- **Cosmological Perturbations**
- **Perturbations of Charged Black Holes**
- **Generalization to Different Twists**

NC gauge theory of gravity

Commutative de Sitter Gauge Theory

- The commutative $SO(4,1)$ de Sitter (DS) gauge theory of gravitation is formulated on a four-dimensional spherically symmetric Minkowski spacetime.
- Metric of Minkowski spacetime:

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

Gauge Fields and Field Strength Tensor:

- DS gauge group: 10-dimensional with gauge fields $h_\mu^A(x)$.
- Gravitational potentials split into:
  - Tetrad fields e_μ^a $(A = a = 0, 1, 2, 3)$
  - Spin connections ω_μ^{ab} $(A = [ab] = [01], [02], [03], ...)$
- Field strength tensor $R_{\mu\nu}^A$ splits into (with $\omega_\mu^{a5} = ke_\mu^a$):
  - Torsion: $R_{\mu\nu}^a \equiv k\,\mathcal{T}_{\mu\nu}^a = k\left[\partial_\mu e_\nu^a - \partial_\nu e_\mu^a + \left(\omega_\mu^{ab}\,e_\nu^c - \omega_\nu^{ab}\,e_\mu^c\right)\eta_{bc}\right]$
  - Curvature: $R_{\mu\nu}^{ab} \equiv F_{\mu\nu}^{ab} = \partial_\mu\omega_\nu^{ab} - \partial_\nu\omega_\mu^{ab} + \left(\omega_\mu^{ac}\omega_\nu^{db} - \omega_\nu^{ac}\omega_\mu^{db}\right)\eta_{cd} + k\left(e_\mu^a e_\nu^b - e_\nu^a e_\mu^b\right)$
- In the limit $k \to 0$, the DS group algebra reduces to the Poincaré Lie algebra.

Commutative de Sitter Gauge Theory

Action: $S = \frac{1}{16\pi G} \int \det(e_\mu^a) R_{\mu\nu}^{ab} e_a^\mu e_b^\nu$

Einstein Equations: $\tilde{R}_\mu^\nu - \frac{1}{2}\tilde{R}\delta_\mu^\nu = 0$

Curvature Tensors:

$$\tilde{R}_{\mu\nu}^{\alpha\beta} = R_{\mu\nu}^{ab} e_a^\alpha e_b^\beta$$

$$\tilde{R}_\mu^\nu = R_{\mu\rho}^{ab} e_a^\nu e_b^\rho = \tilde{R}_{\mu\rho}^{\nu\rho}$$

Ansatz for $SO(4,1)$ Gauge Fields:

$$e_\mu^0 = (A, 0, 0, 0)\,, \qquad e_\mu^1 = \left(0, \frac{1}{A}, 0, 0\right),$$

$$e_\mu^2 = (0, 0, rC, 0)\,, \qquad e_\mu^3 = (0, 0, 0, rC\sin\theta)$$

Spin Connection:

$$\omega_\mu{}^{ab} = \frac{1}{2} e^{\nu a}(\partial_\mu e_\nu{}^b - \partial_\nu e_\mu{}^b) - \frac{1}{2} e^{\nu b}(\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a) - \frac{1}{2} e^{\rho a} e^{\sigma b}(\partial_\rho e_{\sigma c} - \partial_\sigma e_{\rho c}) e_\mu{}^c$$

Solution for A^2: $A^2 = 1 + \frac{\alpha}{r}, \quad \alpha = -2M \Rightarrow A^2 = 1 - \frac{2M}{r}$

Metric and Line Element: $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}, \quad ds^2 = -A^2 dt^2 + \frac{dr^2}{A^2} + r^2 d\Omega^2$

Noncommutative de Sitter Gauge Theory

To formulate a noncommutative gauge theory of gravity, we apply the Seiberg-Witten map to derive corrections in the noncommutative parameter Θ. Let $\hat{\omega}_\mu^{AB}$ denote the nonabelian Yang-Mills field with field strength tensor $\hat{R}_{\mu\nu}^{AB}$.

Field Strength Tensor:

$$\hat{R}_{\mu\nu}^{AB} = \partial_\mu \hat{\omega}_\nu^{AB} - \partial_\nu \hat{\omega}_\mu^{AB} + [\hat{\omega}_\mu, \hat{\omega}_\nu]_\star^{AB}$$

Seiberg-Witten Map: NC gauge fields are expressed in terms of commutative fields to maintain gauge compatibility:

$$\hat{\omega}_\mu^{AB}(\omega) + \delta_{\hat{\lambda}} \hat{\omega}_\mu^{AB}(\omega) = \hat{\omega}_\mu^{AB}(\omega + \delta_\lambda \omega)$$

NC Gauge Transformation:

$$\delta_{\hat{\lambda}} \hat{\omega}_\mu^{AB} = \partial_\mu \hat{\lambda}^{AB} + \eta_{CD} \left(\hat{\omega}_\mu^{AC} \star \hat{\lambda}^{DB} - \hat{\lambda}^{AC} \star \hat{\omega}_\mu^{DB} \right)$$

Recursive Solution:

$$\delta \hat{\omega}_\mu^{AB} = -\frac{i}{4} \Theta^{\nu\rho} \left\{ \hat{\omega}_\nu, \partial_\rho \hat{\omega}_\mu + \hat{R}_{\rho\mu} \right\}_\star^{AB}$$

Expanding $\hat{\omega}_\mu$ up to second order in Θ:

$$\hat{\omega}_\mu = \omega_\mu - i\Theta^{\nu\rho} \omega_{\mu\nu\rho} + \Theta^{\nu\rho} \Theta^{\lambda\tau} \omega_{\mu\nu\rho\lambda\tau} + \mathcal{O}(\Theta^3)$$

First-Order Solution:

$$\hat{\omega}_\mu^{AB} = \omega_\mu^{AB} - \frac{i}{4} \Theta^{\nu\rho} \{ \omega_\nu, \partial_\rho \omega_\mu + R_{\rho\mu} \}^{AB}$$

Noncommutative de Sitter Gauge Theory

NC Correction to Tetrad Fields:

$$\hat{e}^a_\mu(x,\Theta) = e^a_\mu(x) - i\Theta^{\nu\rho} e^a_{\mu\nu\rho}(x) + \Theta^{\nu\rho}\Theta^{\lambda\tau} e^a_{\mu\nu\rho\lambda\tau}(x) + \mathcal{O}(\Theta^3)$$

NC Inverse Tetrads:

$$\widehat{e}^\mu_{\star a} \star \widehat{e}^b_\mu = \delta^b_a$$

$$\widehat{e}^\mu_{\star a}(x,\Theta) = e^\mu_a(x) - i\Theta^{\nu\rho} e^\mu_{a\nu\rho}(x) + \Theta^{\nu\rho}\Theta^{\lambda\tau} e^\mu_{a\nu\rho\lambda\tau}(x) + \mathcal{O}(\Theta^3)$$

NC Spin Connection:

$$\hat{\omega}^{ab}_\mu(x,\Theta) = \omega^{ab}_\mu(x) - i\Theta^{\nu\rho}\omega^{ab}_{\mu\nu\rho}(x) + \Theta^{\nu\rho}\Theta^{\lambda\tau}\omega^{ab}_{\mu\nu\rho\lambda\tau}(x) + \mathcal{O}(\Theta^3)$$

NC Field Tensor:

$$\widehat{R}^{ab}_{\mu\nu} = R^{ab}_{\mu\nu} - i\Theta^{\rho\sigma} R^{ab}_{\mu\nu\rho\sigma} + \Theta^{\rho\sigma}\Theta^{\lambda\tau} R^{ab}_{\mu\nu\rho\sigma\lambda\tau} + O(\Theta^3)$$

NC Scalar Curvature:

$$\hat{R} = \widehat{e}^\mu_{\star a} \star \hat{R}^{ab}_{\mu\nu} \star \widehat{e}^{\nu\dagger}_{\star b}$$

Noncommutative Corrections to Four-Dimensional Non-Rotating Metric

NC Metric Definition: $\hat{g}_{\mu\nu}(x,\Theta) = \frac{1}{2}\eta_{ab}\left(\hat{e}^a_\mu \star \hat{e}^{b\dagger}_\nu + \hat{e}^b_\nu \star \hat{e}^{a\dagger}_\mu\right)$

Star Product: $f \star g = fg - \frac{i}{2}\Theta^{\mu\nu}\partial_\mu f \partial_\nu g + \frac{1}{8}\Theta^{\rho_1\sigma_1}\Theta^{\rho_2\sigma_2}\partial^2_{\rho_1\rho_2} f \partial^2_{\sigma_1\sigma_2} g$

Drinfeld, Abelian, Moyal-Type Twists:

$$\mathcal{F} = \exp\left(i\Theta^{\alpha\beta} V^1_\alpha \otimes V^2_\beta\right)$$

Twists can be categorized as:

- **Killing Twists:** Both V^1_α and V^2_α are Killing vectors.
- **Semi-Killing Twists:** Only one vector is a Killing vector.

Possible vector fields for the twist: $\{\partial_t, \partial_r, \partial_\theta, \partial_\varphi\}$, resulting in six combinations (e.g., $(\partial_r, \partial_\theta)$).

Twist Effects:

- Twists not involving radial coordinates yield consistent physical interpretations.
- Twists involving angular coordinates (φ) introduce off-diagonal metric terms, such as $g_{t\varphi}$.

(t, θ) Twist

Noncommutative Parameter:

$$\Theta^{\mu\nu} = \begin{pmatrix} 0 & 0 & \Theta & 0 \\ 0 & 0 & 0 & 0 \\ -\Theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad [t \overset{\star}{,} \theta] = i\Theta$$

NC-Corrected Schwarzschild Metric:

$$\hat{g}_{00}(x,\Theta) = -\left(1 - \frac{2M}{r}\right) + \frac{M^2(r-2M)(9r-30M)}{48r^6}\Theta^2 + \mathcal{O}(\Theta^3)$$

$$\hat{g}_{11}(x,\Theta) = \left(1 - \frac{2M}{r}\right)^{-1} - \frac{M(11M^2 - 10Mr + 2r^2)}{4r^4(r-2M)}\Theta^2 + \mathcal{O}(\Theta^3)$$

$$\hat{g}_{22}(x,\Theta) = r^2 + \frac{M(11M-4r)(2M-r)}{16r^3}\Theta^2 + \mathcal{O}(\Theta^3)$$

$$\hat{g}_{33}(x,\Theta) = r^2\sin^2\theta + \frac{M(r-2M)(r+M)\sin^2\theta}{4r^3}\Theta^2 + \mathcal{O}(\Theta^3)$$

Properties of the Deformed Spacetime

Symmetry and Horizons:

- Spherical symmetry is broken by the (t, θ) twist.
- Two singularities: $r = 0$ and $r = 2M$ (event horizon, also a Killing horizon).
- The location of the horizon remains unchanged by noncommutative deformation.

Surface Gravity:

$$\kappa^2 = \frac{M^2}{r^4} + \frac{(154M^5 - 117M^4 r + 20M^3 r^2)}{16r^9}\Theta^2 + \mathcal{O}(\Theta^3)$$

At $r = 2M$, the Θ^2 term vanishes, and $\kappa = 1/4M$, implying no NC correction to κ.

Curvature Scalars:

$$R = \frac{3M^2(75M^2 - 47Mr + 7r^2)}{8r^8}\Theta^2 + \mathcal{O}(\Theta^3) \qquad R^{\mu\nu}R_{\mu\nu} = 0$$

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48M^2}{r^6} + \frac{6M^3(129M^2 - 90Mr + 14r^2)}{r^{11}}\Theta^2 + \mathcal{O}(\Theta^3)$$

Next Steps

- Noncommutative Schwarzschild, BTZ, rotating and Reissner-Nordström solutions have been derived.
- Explore scalar and electromagnetic field perturbations in the noncommutative framework.
- Study particle motion and examine related physical properties.
- Addressing conceptual gaps

Thank You!