L_∞-algebras II: Examples

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Outline

- Batalin-Vilkovisky and L_∞
- Yang-Mills
- L_∞ quasi-isomorphisms

Batalin-Vilkovisky and L_∞

BV formalism

- Very general approach to the quantisation of classical field theories
- Needed in cases of open gauge algebras
- Involves doubling the BRST field content
- This doubling introduces a natural symplectic structure $\Rightarrow$ Antibracket
- The BRST operator is extended to a homological vector field that is Hamiltonian wrt this symplectic structure $\Rightarrow$ Master equation

Definition

[hep-th/9206084] Zwiebach; [hep-th/9209099] Lada, Stasheff

An ***L_∞*-algebra** (or homotopy Lie algebra) (L, μ_i) is a differential graded commutative algebra with a set of higher products that are graded totally antisymmetric multilinear maps

$$\mu_i : \underbrace{\mathsf{L} \times \cdots \times \mathsf{L}}_{i\text{-times}} \to \mathsf{L}, \qquad i \in \mathbb{N}_0$$

of degree $2 - i$ which satisfy the homotopy Jacobi identities:

$$\sum_{j+k=n} \sum_{\sigma} \chi(\sigma; l_1, \ldots, l_n)(-1)^k \mu_{k+1}(\mu_j(l_{\sigma(1)}, \ldots, l_{\sigma(j)}), l_{\sigma(j+1)}, \ldots, l_{\sigma(n)}) = 0$$

$$l_i \in \mathsf{L}.$$

Special L_∞-algebras

A **cyclic L_∞-algebra** is an L_∞-algebra with a graded symmetric non-degenerate bilinear pairing

$$\langle \cdot, \cdot \rangle_{\mathsf{L}} : \mathsf{L} \times \mathsf{L} \to \mathbb{R}$$

that satisfies the cyclicity condition:

$$\langle l_1, \mu_i(l_2, \ldots, l_{i+1}) \rangle_{\mathsf{L}} = (-1)^{i+i(|l_1|+|l_{n+1}|)+|l_{i+1}| \sum_{j=1}^{i} |l_j|} \langle l_{i+1}, \mu_i(l_1, \ldots, l_i) \rangle_{\mathsf{L}}; \quad \forall i \in \mathbb{N}.$$

A **tensor product L_∞-algebra** $(\hat{\mathsf{L}}, \hat{\mu}_i)$ is the natural L_∞-algebra induced by the tensor product of an L_∞-algebra (L, μ_i) and a differential graded commutative algebra (A, d).

Since a de Rham complex on a manifold M, $(\Omega^\bullet(M), \mathrm{d})$, is a differential graded commutative algebra its tensor product with an L_∞-algebra (L, μ_i) will again be an L_∞-algebra $(\Omega^\bullet(M, \mathsf{L}), \mu_i')$

Tensored algebras relevant for BV

Gauge algebra	Field theory algebra	BV algebra
L	$\mathsf{L}' = \Omega^\bullet(M, \mathsf{L})$	$\hat{\mathsf{L}} = C^\infty(\mathsf{L}'[1]) \otimes \mathsf{L}'$

Field Theory from L_∞

- Action:

$$S_{\rm BV}[\mathsf{a}] = \sum_{i\geqslant 0} \frac{1}{(i+1)!}\langle \mathsf{a}, \hat{\mu}_i(\mathsf{a},\ldots,\mathsf{a})\rangle_{\hat{\mathsf{L}}}$$

- BRST transformations:

$$Q_{\rm BV}\mathsf{a} = -\sum_{i\geqslant 1} \frac{1}{i!}\hat{\mu}_i(\mathsf{a},\ldots,\mathsf{a})$$

Yang-Mills theory

Setup

- smooth compact 4-dimensional manifold without boundary M
- Lie group G with metric Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \langle\cdot,\cdot\rangle_{\mathfrak{g}})$
- inner product on $\Omega^\bullet(M)$: $(\alpha_1, \alpha_2) = \int_M \alpha_1 \wedge *\alpha_2 \Rightarrow$ induces product on $\Omega^\bullet(M, \mathfrak{g})$

Second order L_∞ description

Spaces:

$$\begin{array}{cccc} c\in & A\in & A^\dagger\in & c^\dagger\in \\ \Omega^0(M,\mathfrak{g}) \xrightarrow{\mu_1=d} & \Omega^1(M,\mathfrak{g}) \xrightarrow{\mu_1=d*d} & \Omega^3(M,\mathfrak{g}) \xrightarrow{\mu_1=d} & \Omega^4(M,\mathfrak{g}) \end{array}$$

Non-vanishing maps:

$$\mu_1(c)=dc,\quad \mu_1(A)=d*dA,\quad \mu_1(A^\dagger)=dA^\dagger,$$

$$\mu_2(A_1,A_2)=d*[A_1,A_2]+[A_1,*dA_2]+[A_2,*dA_1],$$

all other possible μ_2 are just Lie brackets,

$$\mu_3(A_1,A_2,A_3)=[A_1,*[A_2,A_3]]+[A_2,*[A_3,A_1]]+[A_3,*[A_1,A_2]]$$

Action

$$S_{\mathrm{BV}}[\mathsf{a}] = \sum_{i\geqslant 0} \frac{1}{(i+1)!}\langle \mathsf{a}, \hat{\mu}_i(\mathsf{a},\dots,\mathsf{a})\rangle_{\hat{\mathsf{L}}}, \qquad \mathsf{a} = c + A + A^\dagger + c^\dagger$$

μ_1:

$$\int_M \langle A, d * dA\rangle = \int_M \langle dA, *dA\rangle$$

μ_2:

$$\int_M \langle A, d*[A,A] + 2[A,*dA]\rangle = 3\int_M \langle dA, *[A,A]\rangle$$

μ_3:

$$\int_M \langle A, 3[A,*[A,A]]\rangle = 3\int_M \langle [A,A], *[A,A]\rangle$$

$$S_{\text{classic}} = \int_M \frac{1}{2}\langle F, *F\rangle \qquad F = dA + \frac{1}{2}[A,A]$$

All other possible combinations with non-vanishing pairings are:

$$\langle c^\dagger, \mu_2(c,c)\rangle, \langle A^\dagger, \mu_1(c)\rangle, \langle A^\dagger, \mu_2(A,c)\rangle, \langle A, \mu_2(A^\dagger, c)\rangle, \langle c, \mu_1(A^\dagger)\rangle, \langle c, \mu_2(c, c^\dagger)\rangle.$$

Combining into:

$$S_{\mathrm{BV}} = S_{\text{classic}} + \langle A^\dagger, \mu_1(c)\rangle + \langle A^\dagger, \mu_2(A,c)\rangle + \frac{1}{2}\langle c^\dagger, \mu_2(c,c)\rangle$$

BRST transformations: $Q_{\mathrm{BV}}\mathsf{a} = -\sum_{i\geqslant 1}\frac{1}{i!}\hat{\mu}_i(\mathsf{a},\ldots,\mathsf{a})$.
For example take the ghost zero component:

$$-Q_{\mathrm{BV}}A^\dagger + \mathcal{O}(c^\dagger, A^\dagger) = d{*}dA + \frac{1}{2}d{*}[A,A] + [A,{*}dA] + \frac{1}{2}[A,{*}[A,A]] = d{*}F + [A,{*}F]$$

First order L_∞ description

Spaces:

$$\Omega^0(M,\mathfrak{g}) \xrightarrow{\mu_1=d} \Omega^2_+(M,\mathfrak{g})\oplus\Omega^1(M,\mathfrak{g}) \xrightarrow{\mu_1=(1+P_+)+d} \Omega^2_+(M,\mathfrak{g})\oplus\Omega^3(M,\mathfrak{g}) \xrightarrow{\mu_1=0+d} \Omega^4(M,\mathfrak{g})$$

Non-vanishing maps:

$$\mu_1(c) = dc, \quad \mu_1(A+B) = (P_+B + P_+dA) + dP_+B, \quad \mu_1(A^\dagger) = dA^\dagger,$$

$$\mu_2(A_1+B_1, A_2+B_2) = P_+[A_1,A_2] + [A_1,B_2] + [A_2,B_1],$$

$$\mu_2(A+B, A^\dagger + B^\dagger) = [A,A^\dagger] + [B,B^\dagger]$$

all other possible μ_2 are just Lie brackets.

Here $*$ induces the decomposition $\Omega^2(M,\mathfrak{g}) = \Omega^2_+(M,\mathfrak{g}) \oplus \Omega^2_-(M,\mathfrak{g})$ via the projectors $P_\pm = \frac{1}{2}(1 \pm *)$: $\Omega^2_\pm(M,\mathfrak{g}) = P_\pm\Omega^2(M,\mathfrak{g})$

Action

μ_1:

$$\int_M \langle A+B_+, \mu_1(A+B_+)\rangle = \int_M \langle A+B_+, B_+ + P_+ dA + dB_+\rangle = \int_M 2\langle dA, B_+\rangle + \langle B_+, B_+\rangle$$

μ_2:

$$\int_M \langle A+B_+, \mu_2(A+B_+, A+B_+)\rangle = \int_M \langle A+B_+, P_+[A,A] + 2[A, B_+]\rangle = \int_M 3\langle B_+, [A,A]\rangle$$

$$S_{\text{classic}} = \int_M \langle F, B_+\rangle + \frac{1}{2}\langle B_+, B_+\rangle$$

Integrating out B_+ this action is classically equivalent to the second order formulation.

Morphism

We say a collection of multilinear, totally graded antisymmetric homogeneous maps $\phi_i : \mathsf{L}^{\times i} \to \mathsf{L}'$ of degree $1-i$, $i \in \mathbb{N}$, is an L_∞-morphism between two L_∞-algebras (L, μ) and (L', μ') if they satisfy:

$$\sum_{j+k=n} \sum_{\sigma \in \mathrm{Sh}(j;n)} (-1)^k \chi(\sigma; l_1, \ldots, l_n) \phi_{k+1}(\mu_j(l_{\sigma(1)}, \ldots, l_{\sigma(j)}), l_{\sigma(j+1)}, \ldots, l_{\sigma(n)}) =$$

$$= \sum_{k_1+\cdots+k_j=n} \frac{1}{j!} \sum_{\sigma \in \mathrm{Sh}(k_1,\ldots,k_{j-1};n)} \chi(\sigma; l_1, \ldots, l_n) \zeta(\sigma; l_1, \ldots, l_n) \times$$

$$\times \mu'_j(\phi_{k_1}(l_{\sigma(1)}, \ldots, l_{\sigma(k_1)}), \ldots, \phi_{k_j}(l_{\sigma(k_1+\cdots+k_{j-1}+1)}, \ldots, l_{\sigma(n)}))$$

If ϕ_1 induces an isomorphism of cohomologies $H^\bullet_{\mu_1}(\mathsf{L}) \cong H^\bullet_{\mu'_1}(\mathsf{L}')$ it is called a quasi-isomorphism. Quasi-isomorphisms correspond to physically equivalent systems.

Simple example for scalar fields

Take two actions:

$$S = \int_M d^4x \left(\tfrac{1}{2}\varphi(-\Box - m^2)\varphi - \tfrac{\lambda}{4!}\varphi^4 \right)$$

$$S' = \int_M d^4x \left(\tfrac{1}{2}\varphi(-\Box - m^2)\varphi + \tfrac{1}{2}X^2 + \tfrac{1}{2}\sqrt{\tfrac{\lambda}{3}}X\varphi^2 \right),$$

that are classically equivalent via the eom $X + \frac{1}{2}\sqrt{\frac{\lambda}{3}}\varphi^2 = 0$.

L_∞ for scalar fields

- L

$$\to C^\infty(M) \to C^\infty(M) \to$$

$$\mu_1(\varphi) = (-\Box - m^2)\varphi \qquad \mu_3(\varphi_1, \varphi_2, \varphi_3) = -\lambda\varphi_1\varphi_2\varphi_3$$

- L′

$$\to C^\infty(M) \oplus C^\infty(M) \to C^\infty(M) \oplus C^\infty(M) \to$$

$$\mu_1(\varphi+X) = (-\Box-m^2)\varphi+X \qquad \mu_2(\varphi_1+X_1, \varphi_2+X_2) = \sqrt{\tfrac{\lambda}{3}}(X_1\varphi_2+X_2\varphi_1+\varphi_1\varphi_2)$$

The chain map ϕ_1: $\phi_1(\varphi + X) = \varphi$ and $\phi_1(\zeta + Y) = \zeta$ does not affect the vector space cohomology since the addition of the identity map to the differential makes no difference to $H^\bullet(\mathrm{L})$.

It remains to show that ϕ is an L_∞-morphism. The choice for the non-vanishing components to be just: $\phi_2(\varphi_1 + X_1, \zeta_1 + Y_1) = \sqrt{\frac{\lambda}{3}}\varphi_1 Y_1$ can be easily show to satisfy the conditions above.

Quasi-isomorphism in YM

- One can show the two formulations of Yang-Mills have the same cohomology complex
- Then one can construct an L_∞-morphism between the two formulations that states (in the coalgebra picture where the requirement analogous to the conditions above is $Q_{\mathrm{BV}} \circ \Phi = \Phi \circ Q'_{\mathrm{BV}}$):

$$\Phi(c) = c \qquad \Phi(A) = A \qquad \Phi(B_+) = -F_+$$
$$\Phi(A^\dagger) = A^\dagger \qquad \Phi(B_+^\dagger) = 0 \qquad \Phi(c^\dagger) = c^\dagger$$

Summary

Summary

- Connection of L_∞ structures and field theory/Batalin-Vilkovisky formalism
- Yang-Mills both in first and second order formalism as L_∞ theories
- The physical meaning of the equivalence classes induced by L_∞ quasi-isomorphisms
- Classical equivalence of Yang-Mills and scalar field theory formulations as L_∞ quasi-isomorphic theories
- It is important to notice one does not need cyclic L_∞-algebras to construct quasi-isomorphisms indicating one can find equivalent theories even if one does not have an action functional description