Hamiltonian Dynamics of Dissipative Systems

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Introduction

Hamiltonian formulation of classical mechanics is very useful since it gives a nice geometric description of time evolution and provides a natural way to extend a classical theory to quantum theory.

Originally, the Hamiltonian formulation was given only for isolated systems, and later extended to systems with time-dependent potentials, but no dissipative forces.

There have been many attempts to introduce dissipation into Hamiltonian formulation. Here we present one that arises quite naturally as a geometric generalization of the known cases.

Non-dissipative time-independent systems

The dynamics of isolated (non-dissipative and time-independent) systems can be given in terms of a Hamiltonian function defined on a phase space M. The phase space is $2n$-dimensional symplectic manifold with a symplectic form ω (non-closed and non-degenerate ($\omega^n \neq 0$)).

Darboux theorem:

> For every point of a $2n$-dimensional symplectic manifold there exists a neighbourhood on which it is possible to choose a coordinate system $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ in which the symplectic form can be expressed as:
>
> $$\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i \,.$$

Since ω is closed, we can locally define a canonical 1-form α such that $\omega = \mathrm{d}\alpha$. In Darboux coordinates it has the form:

$$\alpha = p_i \, \mathrm{d}q^i \,.$$

For a Hamiltonian function $H \in C^\infty(M)$ we can define a Hamiltonian vector field $X_H \in \Gamma(TM)$ as:

$$\iota_{X_H}\omega = -\,\mathrm{d}H\,,$$
$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}\,.$$

The trajectories of the system are the integral curves of this vector field which in Darboux coordinates gives the Hamilton equations of motion:

$$\dot{q}^i = \frac{\partial H}{\partial p_i}\,, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}\,.$$

Every symplectic manifold is also a Poisson manifold since the Poisson bivector $\mathcal{P} \in \Gamma(TM \wedge TM)$ can be defined as:

$$\iota_{\mathcal{P}(\xi)}\omega = \xi \,, \forall \xi \in \Gamma(T^*M) \,,$$
$$\mathcal{P} = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} \,.$$

This defines a Poisson bracket of two functions $f, g \in C^\infty(M)$:

$$\{f, g\} = \mathcal{P}(f, g) \,.$$

The Jacobi identity of the Poisson bracket translates to the following property of the Poisson bivector:

$$[\mathcal{P}, \mathcal{P}]_\mathsf{S} = \mathcal{P}^{\rho\sigma}\partial_\rho \mathcal{P}^{\mu\nu}\partial_\mu \wedge \partial_\nu \wedge \partial_\sigma = 0 \,.$$

Canonical transformations:

Diffeomorphism transformations that leave symplectic form invariant.

Locally, they look like a change in Darboux coordinates. Since the symplectic is invariant, a canonical 1-form is shifted by a closed 1-form:

$$p_i \, \mathrm{d}q^i = P_i \, \mathrm{d}Q^i + \mathrm{d}F_1(q, Q) \,,$$
$$p_i = \frac{\partial F_1}{\partial q^i} \,, \quad P_i = -\frac{\partial F_1}{\partial Q^i} \,.$$

Non-dissipative time-dependent systems

We extend the phase to include time explicitly so that the manifold of interest is $M_E = M \times \mathbb{R}$.

The canonical 1-form is extended into the Poincaré-Cartan 1-form:

$$\eta = p_i \,\mathrm{d}q^i - H(q,p,t)\,\mathrm{d}t\,,$$
$$\mathrm{d}\eta = \omega - \mathrm{d}H \wedge \mathrm{d}t\,.$$

Hamiltonian vector fields:

$$\iota_{X_H^E}\,\mathrm{d}\eta = 0\,,$$
$$X_H^E = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} + \frac{\partial}{\partial t}\,.$$

Canonical transformations:

$$p_i\,\mathrm{d}q^i - H\,\mathrm{d}t = P_i\,\mathrm{d}Q^i - K\,\mathrm{d}t + \mathrm{d}F_1(q,Q,t)\,,$$
$$p_i = \frac{\partial F_1}{\partial q^i}\,, \quad P_i = -\frac{\partial F_1}{\partial Q^i}\,, \quad K = H + \frac{\partial F_1}{\partial t}\,.$$

What is the geometrical structure of M_E?

In symplectic case, ω was closed and degenerate. Closure of ω directly translates to the closure of $\mathrm{d}\eta$, which is trivial.

Non-degeneracy of ω ($\omega^n \neq 0$) can also be generalized to:

$$\eta \wedge \mathrm{d}\eta^n = -H\omega^n \wedge \mathrm{d}t\,.$$

The existence of such 1-form makes M_E the so called contact manifold.

Example: 1D dissipative system

Consider a particle in 1 dimension in a potential V and a friction force proportional to the velocity of a particle. The equation of motion has the form:

$$m\ddot{q} = -\frac{\partial V}{\partial q} - m\gamma q \, .$$

We can rewrite this second-order equation as a system of two first-order equations in order to get the same form as the Hamilton's equations of motion:

$$\dot{q} = \frac{p}{m} \, ,$$

$$\dot{p} = -\gamma p - \frac{\partial V}{\partial q} \, .$$

Caldirola-Kanai coordinates:

$$q_{CK} = q\,, \quad p_{CK} = \mathrm{e}^{\gamma t} p\,.$$

Equations of motion:

$$\dot{q}_{CK} = \frac{p_{CK}}{m}\mathrm{e}^{-\gamma t}\,,$$
$$\dot{p}_{CK} = -\frac{\partial V}{\partial q_{CK}}\mathrm{e}^{\gamma t}\,.$$

In these new coordinates there exists a Hamiltonian function that generates these equations of motion:

$$H(q_{CK}, p_{CK}, t) = \frac{1}{2m} p_{CK}^2 \mathrm{e}^{-\gamma t} + V(q_{CK})\mathrm{e}^{\gamma t}\,.$$

We introduced the change in coordinates that transformed a dissipative time-independent system into a non-dissipative time-dependent system, but this transformation was not canonical.

Dissipative time-independent systems

Let $\mathcal{T}$ be a $(2n+1)$-dimensional contact manifold with a contact form η.

The generalization of the Darboux theorem for contact manifolds states that it is locally possible to choose coordinates $(q_1, \dots, q_n, p_1, \dots, p_n, S)$ such that the contact form takes the form:

$$\eta = \mathrm{d}S - p_i \, \mathrm{d}q^i \, .$$

The contact form defines an isomorphism $g : \Gamma(TM) \to \Gamma(T^*M)$:

$$g(v) = \iota_v \, \mathrm{d}\eta + (\iota_v \eta)\eta \, ,$$
$$g = \frac{1}{2}\eta \vee \eta - \omega \, .$$

This isomorphism defines the so called Reeb vector field $V \in \Gamma(TM)$:

$$V = g^{-1}(\eta) = \frac{\partial}{\partial S} \, .$$

This is equivalent to the statement that:

$$\iota_V \eta = 1 \, , \quad \iota_V \, \mathrm{d}\eta = 0 \, .$$

Contact transformations:

The change in coordinates $(q, p, S) \to (Q, P, S')$ that leave contact form invariant, up to a multiplication function:

$$\eta \to f\eta\,,$$
$$\mathrm{d}S' - P_i\,\mathrm{d}Q^i = f(\,\mathrm{d}S - p_i\,\mathrm{d}q^i)\,.$$

Here, $S'(q, Q, S)$ can be used as a generating function:

$$f = \frac{\partial S'}{\partial S}\,, \quad fp_i = -\frac{\partial S'}{\partial q^i}\,, \quad P_i = \frac{\partial S'}{\partial Q^i}\,.$$

The special case when $f = 1$:

$$S'(q, Q, S) = S - F_1(q, Q)$$

corresponds to a canonical transformation.

Hamiltonian vector field $X_{\mathcal{H}} \in \Gamma(T\mathcal{T})$ for a contact Hamiltonian function $\mathcal{H} \in C^{\infty}(\mathcal{T})$:

$$\mathcal{L}_{X_{\mathcal{H}}}\eta = f\eta\,, \quad \mathcal{H} = -\iota_{X_{\mathcal{H}}}\eta\,.$$

Equivalently, these equations can be rewritten as:

$$X_{\mathcal{H}} = g^{-1}(\,\mathrm{d}\mathcal{H} + (f - \mathcal{H})\eta)\,.$$

The function f that appears here is not arbitrary, but it turns out to be:

$$f = -\iota_V\,\mathrm{d}\mathcal{H} = -\frac{\partial \mathcal{H}}{\partial S}\,.$$

The Hamiltonian vector field in Darboux coordinates takes the form:

$$X_{\mathcal{H}} = \frac{\partial \mathcal{H}}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial \mathcal{H}}{\partial q^i} + p_i\frac{\partial \mathcal{H}}{\partial S}\right)\frac{\partial}{\partial p_i} + \left(p_i\frac{\partial \mathcal{H}}{\partial p_i} - \mathcal{H}\right)\frac{\partial}{\partial S}$$

The trajectories of the system are just integral curves of the Hamiltonian vector field which gives us the equations of motion:

$$\dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i}\,,$$
$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i} - p_i \frac{\partial \mathcal{H}}{\partial S}\,,$$
$$\dot{S} = p_i \frac{\partial \mathcal{H}}{\partial p_i} - \mathcal{H}\,.$$

The time derivative of a general function along a trajectory is then equal to:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial q^i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q^i} - p_i \frac{\partial F}{\partial p_i}\frac{\partial \mathcal{H}}{\partial S} + p_i \frac{\partial F}{\partial S}\frac{\partial \mathcal{H}}{\partial p_i} - \mathcal{H}\frac{\partial F}{\partial S}$$

Instead of the Poisson structure, contact manifolds admit a Jacobi structure. A Jacobi manifold is a manifold equipped with a Jacobi bivector $\pi \in \Gamma(T\mathcal{T} \wedge T\mathcal{T})$ and a vector field $V \in \Gamma(T\mathcal{T})$ such that:

$$[\pi, \pi]_\mathsf{S} = \pi^{\rho\sigma}\partial_\rho \pi^{\mu\nu}\partial_\mu \wedge \partial_\nu \wedge \partial_\sigma = 2V \wedge \pi \,,$$
$$[\pi, V]_\mathsf{S} = \left(\frac{1}{2}V^\sigma \partial_\sigma \pi^{\mu\nu} + \pi^{\sigma\mu}\partial_\sigma V^\nu\right)\partial_\mu \wedge \partial_\nu = 0 \,.$$

The vector V here is just the Reeb vector field, while the Jacobi bivector can be defined through:

$$\iota_{\pi(\xi)}\eta = 0 \,, \forall \xi \in \Gamma(T^*\mathcal{T}) \,,$$
$$\iota_{\pi(\xi)}\, \mathrm{d}\eta = -\xi + (\iota_V \xi)\eta \,, \forall \xi \in \Gamma(T^*\mathcal{T}) \,.$$

In Darboux coordinates:

$$\pi = \left(\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial S}\right) \wedge \frac{\partial}{\partial p_i} \,.$$

The Hamiltonian vector field can be written in terms of Jacobi bivector and Reeb vector field:

$$X_{\mathcal{H}} = -\pi(\mathcal{H}, \cdot) - \mathcal{H} V ,$$

while the time evolution of arbitrary function along a trajectory is determined by a Jacobi bracket:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, \mathcal{H}\}_{\mathrm{J}} - \mathcal{H} V(F) ,$$

where the Jacobi bracket of two functions is defined as:

$$\{f, g\}_{\mathrm{J}} = \pi(f, g) .$$

Note that the Jacobi bracket does not satisfy the Jacobi identity. Instead we have:

$$\{\{f_1, f_2\}_{\mathrm{J}}, f_3\}_{\mathrm{J}} + \text{c.p.} = -\frac{1}{2} \left(\{f1, f2\}_{\mathrm{J}} \iota_V \, \mathrm{d}f_3 + \text{c.p.} \right)$$

Examples

Separable contact Hamiltonians:

$$\mathcal{H}(q,p,S) = H(q,p) + h(S)\,.$$

This kind of Hamiltonians give friction forces proportional to velocities.

The function h controls the time evolution of mechanical energy:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -p_i \frac{\partial H}{\partial p_i}\frac{\partial h}{\partial S}$$

1-dimensional particle with a friction force proportional to the square of the velocity:

$$\mathcal{H}(q,p,S) = \frac{1}{2m}\left(p + 2\gamma S\right)^2 + \mathrm{e}^{-2\gamma q}\int^q \mathrm{e}^{2\gamma q'}\frac{\partial V(q')}{\partial q'}\,\mathrm{d}q'$$

Dissipative time-dependent systems

We extend the contact phase space $\mathcal{T}$ into $\mathcal{T}_E = \mathcal{T} \times \mathbb{R}$ to include time dependence. We also extend the contact 1-form into:

$$\eta_E = \mathrm{d}S - p_i \, \mathrm{d}q^i + \mathcal{H}(q, p, S, t) \, \mathrm{d}t$$

Hamiltonian vector field:

$$\mathcal{L}_{X^E_{\mathcal{H}}} \eta_E = f \eta_E \,, \quad \iota_{X^E_{\mathcal{H}}} \eta_E = 0 \,,$$
$$X^E_{\mathcal{H}} = X_{\mathcal{H}} + \frac{\partial}{\partial t}$$

Equations of motion have the same form as before, just with addition of $\dot{t} = 1$ which tells us that t can be used as a parameter on the trajectory.

Time-dependent contact transformations:

$$f(\,\mathrm{d}S - p_i\,\mathrm{d}q^i + \mathcal{H}\,\mathrm{d}t) = \,\mathrm{d}S' - P_i\,\mathrm{d}Q^i + \mathcal{K}\,\mathrm{d}t$$

$$f = \frac{\partial S'}{\partial S}\,, \quad fp_i = -\frac{\partial S'}{\partial q^i}\,, \quad P_i = \frac{\partial S'}{\partial Q^i}\,, \quad \mathcal{K} = f\mathcal{H} - \frac{\partial S'}{\partial t}$$

Caldirola-Kanai transformation is an example of the time-dependent contact transformation:

$$\mathcal{H}(q,p,S,t) = \frac{p^2}{2m} + V(q) + \gamma S\,,$$

$$(q,p,S,t) \to (q_{CK} = q, p_{CK} = \mathrm{e}^{\gamma t}p, S' = \mathrm{e}^{\gamma t}S, t)\,,$$

$$\mathcal{K} = \mathrm{e}^{\gamma t}(\mathcal{H} - \gamma S) = \frac{p_{CK}^2}{2m}\mathrm{e}^{-\gamma t} + V(q_{CK})\mathrm{e}^{\gamma t}\,.$$

The geometric structure of the extended contact phase space:

$$(\,\mathrm{d}\eta)^{n+1} \neq 0 \to \text{symplectic manifold}$$

Conclusions and Outlook

- Any classical system can be described in terms of the Hamiltonian formulation.
- There is a geometric description for the dynamics of any classical system.

- While this formulation gives a geometric description for any classical system, it is still unclear what role does it play in the quantization of dissipative systems.