# Open problems in mathematical physics

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# Motivation

- Allen and Lidstrom: "*Life, the Universe, and everything: 42 fundamental questions*" Phys. Scr. 92 (2017) 012501

  -problems from CCP, dark energy, QG, BHIP, GUT, phases of matter,..., beginning of life and consciences.

- A. Coley: "*Open problems in mathematical physics*" Phys. Scr. 92 (2017) 093003

  -review of various problems in mathematics, theoretical and mathematical physics...wich with a list of 42 open problems in MP.

# Historical note

Importance of posing problems/challenges (publicly):

- 1530. contest in solving cubic equations (Niccol Tartaglia and Antonio Fiore)

- 1696. John Bernoulli posed a challenging problem:
  "*To find the curve connecting two points, at different heights and not on the same vertical line, along which a body acted upon only by gravity will fall in the shortest time.*"-brachistochrone

  -this can be solved only if one knows calculus, and the winners of the contes were: Leibniz, l'Hospital (Leibniz's student) and anonymous winner ("we know the lion by his claw").

# Overview of famous lists of problems

Hilbert problems (1900)

- list of 23 problems (6 concerning physics) that range over a number of topics in mathematics of the time
- huge impact on the development of mathematics in 20th century
- 4 unsolved problems (H6, H8, H12 and H16)

Smale's problems (1998)

- list of 18 problems inspired by the original Hilbert's list and behest of Vladimir Arnold
- 9 problems remain unsolved

The Millennium Prize problems (2000)

- 7 problems with 1 million USD prize (only one was solved in 2003 by G. Perelman: "The Poincare conjecture")
- includes 2 problems in physics (YM and Navier-Stokes)

# Mathematical Physics (what is it?)

⇒ The Journal of Mathematical Physics defines the field as "*the application of mathematics to problems in physics and the development of mathematical methods suitable for such applications and for the formulation of physical theories*". It is a branch of applied mathematics, but deals with physical problems.

⇒ The term "mathematical physics" is sometimes used to denote research aimed at studying and solving problems inspired by physics or thought experiments within a mathematically rigorous framework. In this sense, mathematical physics covers a very broad academic realm distinguished only by the blending of pure mathematics and physics. Although related to theoretical physics, mathematical physics in this sense emphasizes the mathematical rigor of the same type as found in mathematics.

⇒ Faddeev, L. D. (2000), "*Modern Mathematical Physics: what it should be*?", arXiv:math-ph/0002018

# Problems in Mathematical Physics

We will define them as:

- problems that are well-formulated (i.e. well-posed) mathematical problems which are of interest to physicists.
- and will not include problems where the basic underlying physics is not understood and explicit well-posed mathematical problem can not be formulated. (like QG)
- will not include pure math problems (like Riemann hypothesis)

# Famous lists of problems in Math.Phys.

Simon's problems (1984)

- 15 open problems that are well-formulated and of interest to physicists.
- Simon's problems (2000):15 problems on Schrodinger operators

Yau's problems (1982)

- 120 open questions (mostly Diff.Geom.-like)

Penrose problems (1982)

- 14 unsolved problems in GR

Bartnik problems (1989)

- 53 open problems in mathematical GR

Coley's problems (2017)

- 42 open problems in Math.Phys.

Some Math.Phys. problems from the lists

# Problem 1

$\boxed{\textit{Mathematical treatment of the axioms of physics.}}$

In particular

- the axiomatic treatment of probability with limit theorems for the foundation of statistical physics
- the rigorous theory of limiting processes "which lead from the atomistic view to the laws of motion of continua."

This is the famous H6!

# Problem 2

$\boxed{\textit{Does } P = NP\textit{?}}$

- It asks whether every problem whose solution can be quickly verified (technically, verified in polynomial time) can also be solved quickly (again, in polynomial time).
- An answer to the $P = NP$ question would determine whether problems that can be verified in polynomial time can also be solved in polynomial time. If it turned out that $P \neq NP$, it would mean that there are problems in NP that are harder to compute than to verify: they could not be solved in polynomial time, but the answer could be verified in polynomial time.

On the Smale's and Millenium Prize List!

# Problem 4

*Yang–Mills existence and mass gap.*

- establish the existence of the quantum Yang–Mills theory and a mass gap rigorously.
- As a classical field theory, its solutions propagate at the speed of light and so its quantum version describes massless gluons.
- The so-called mass gap is the problem that color confinement only allows bound states of gluons, which form massive particles.

On the Millenium Prize List, Simons and Yau's list!

# Problem 5

*Existance for Newtonian gravitating particles.*

- prove that the set of initial conditions which fails to have global solutions is of measure zero.
- existence of noncollisional singularities in the Newtonian N-body problem.

On the Simons(1984) list!

# Problem 6

*Develop a comprehensive theory of the long time behavior of dynamical systems including a theory of the onset of, and of fully developed, turbulence.*

On the Simons(1984) list!

# Problems 7 and 8

*Prove that any YM field on $S^4$ is either self-dual or antiself-dual.*

*Prove that the moduli space of the self-dual fields on $S^4$ with a fixed Pontryagin number is connected.*

On the Yau's list!

# Problem 9

*Formulation of the renormalization group and proof of universality.*

- develop a mathematically precise version of the renormalization transformations for n-dimensional Ising-type systems.
- in particular, show that the critical exponents in the three-dimensional Ising models with nearest neighbor coupling but different bond strengths in the three directions are independent of the ratios of these bond strengths.

The "renormalization group theory" of critical phenomena is often claimed to "explain" universality (rather than universality being assumed), but the original Wilson theory is on functions of infinitely many variables and it is far from clear how to formulate the maps in a mathematically precise way (let alone then analyze their fixed point structure).
On the Simons(1984) list!

# Problem 10

*Problems in QFT*

- give a precise mathematical construction of quantum chromodynamics, the model of strong interaction physics.
- construct any non-trivial renormalizable but not super-renormalizable quantum field theory.
- prove that quantum electrodynamics is not a consistent theory.
- prove that a non-trivial lattice cutoff theories theory does not exist.

The problem is whether QFT really is a mathematical theory at all. This question remains open for any nonlinear quantum field theory in three-space plus one-time dimensions. The basic difficulty in formulating the mathematical problem is the singular nature of the nonlinear equations proposed. Physicists eventually developed sets of ad hoc rules to cancel the infinities in QFT and to calculate observable effects to an astonishing accuracy (in QED).

# Problem 11

*Validity of approximations in GR*

- Prove an "approximate solution" result for the (vacuum) Einstein equations in some suitable norm that would provide a good way to evaluate approximate/ asymptotic and numerical solutions.
- Show that a solution of the linearized (about Minkowski space) Einstein equations is close to a (nonflat) exact solution.
- Determine the range of validity of the post- Newtonian and post-Minkowskian asymptotic expansions.
- Prove rigorously the existence of a limit in which solutions of the Einstein equations reduce to Newtonian spacetimes.

Problems 18-21 on Bartnik's list.

# Problem 12

*Causality and singularity in GR*

- Is there a timelike geodesically complete inextendible Lorentz manifold satisfying an energy condition and having a partial Cauchy surface which contains a trapped surface?
- Show that a weak Cauchy surface in a globally hyperbolic spacetime satisfying suitable energy conditions cannot contain an inextendible null geodesic.
- Prove that a 'cosmological spacetime' satisfying the timelike convergence condition is either timelike geodesically incomplete or it splits as $R \times M^3$ isometrically (and is thus static).
- Prove a singularity theorem assuming the dominant energy condition rather than the timelike convergence condition.
- Determine the weakest condition on the smoothness of the metric in the initial value problem for maximizing geodesics to have a unique solution.

Problems 30-34 on Bartnik's list.

# Problems 13 and 14

*Show that there is no vacuum equilibrium configuration involving more than one black hole.*

*Find an exact solution of the Einstein equations which represents two orbiting bodies. Is the 2-body system unstable in Einstein gravity?*

On the Penrose and Bartnik's list!

# Problem 15

*Prove the weak cosmic censorship conjecture.*

- The weak cosmic censorship hypothesis was conceived by Roger Penrose in 1969 and posits that no naked singularities exist in the universe.
- The weak cosmic censorship hypothesis asserts there can be no singularity visible from future null infinity. In other words, singularities need to be hidden from an observer at infinity by the event horizon of a black hole.
- Mathematically, the conjecture states that, for generic initial data, the maximal Cauchy development possesses a complete future null infinity.

# Problem 16

*Find a proof of the Penrose inequality or present a counterexample in the general case.*

- This inequality estimates the mass of a spacetime in terms of the total area of its black holes and is a generalization of the positive mass theorem
- Violation of this inequality would constitute a counterexample to weak cosmic censorship, while a proof of this inequality would provide evidence in favor of weak cosmic censorship.
- In fact, such a proof would possibly lead to an approach for attacking the cosmic censorship conjecture using methods in partial differential equations

# Problem 17

*Prove the stability of the Kerr-Newman black hole.*

# Problem 18

*Determine when a 4D spacetime can be explicitly constructed from its scalar curvature invariants and determine the minimal set of such invariants.*

# Problem 19

*Prove that test particles move on spacetime geodesics.*

- Geodesic hypothesis: One of the postulates of GR is that point particles with negligible mass will travel along geodesics of the spacetime.
- The main problem is how to make the process of "taking the negligible mass limit" rigorous.

# Problems 20 and 21

$\boxed{\textit{Determine the uniqueness of black holes in higher dimensions.}}$

$\boxed{\textit{Determine the stability of higher dimensional black holes.}}$

# Problem 22

*Prove a cosmic no-hair theorem in generic inhomogeneous space-times.*

# Problem 23

*Provide a rigorous mathematical definition for averaging in GR.*

- The gravitational field equations on large scales are obtained by averaging or coarse graining the Einstein field equations of GR.
- The averaging problem in cosmology is crucial for the correct interpretation of cosmological data.
- The so-called fitting problem is perhaps the most important unsolved problem in mathematical cosmology.

# Fundamental problems that may lead to problems in Math.Phys.

$\boxed{\textit{Resolve the black hole information paradox.}}$

$\boxed{\textit{The cosmological constant problem and dark energy.}}$

$\boxed{\textit{Formulate a fully consistent theory of QG.}}$

At this point it is perhaps illuminating to recall the quote by Werner von Braun who said that:

"*Basic research is what I am doing when I do not know what I am doing*".

Thank you for your attention :)