Frustrated Quantum 1D Systems

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Qualifying Exam
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April 12, 2019

Outline

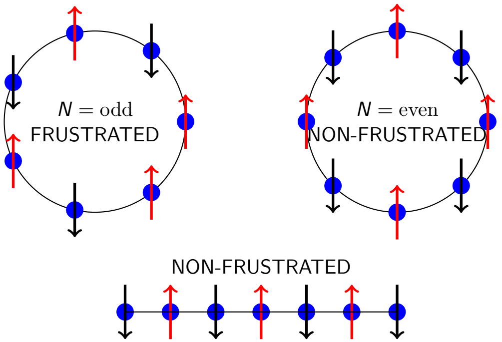
- Frustration
- Magnetization in the frustrated XY chain
  - symmetries, the exact solution, computing the magnetization
  - results: magnetization and the two-point correlation function, quantum phases
- Frustration and Cluster Ising models
- Conclusion

Frustration

- competition of different interactions that cannot simultaneously minimize the energy of the system

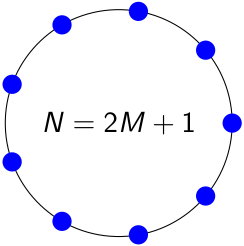
Geometrical Frustration

- Ising antiferromagnet on non-bipartite lattices



How to frustrate geometrically 1D (quantum) systems?

- Interactions that favor staggered order (e.g. antiferromagnetic)
- Odd number of lattice sites N
- Periodic boundary conditions



- Quantum systems: the energy contributions of the non-commuting terms in the Hamiltonian cannot be simultaneously minimized

$$H = \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x + \sum_{j=1}^{N} \sigma_j^y \sigma_{j+1}^y$$

Quantum Mechanics + Geometrical Frustration → new properties?

Quantum Ising chain in a magnetic field:

[Dong et al J. Stat. Mech. 2016]
frustration induces a low-energy gapless spectrum above the ground state with a peculiar longitudinal spin–spin correlations, decaying linearly with distance

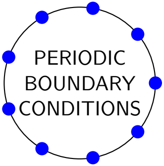
[Giampaolo et al. arXiv 2018]
violation of the area law for the entanglement entropy

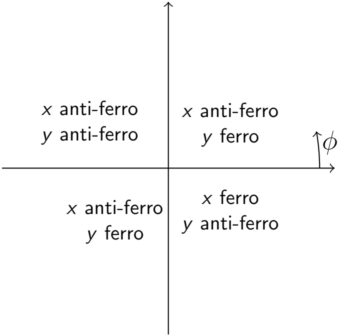
Line of research

- solving frustrated quantum integrable models
- examining the zero-temperature properties: effects of frustration on quantum phases

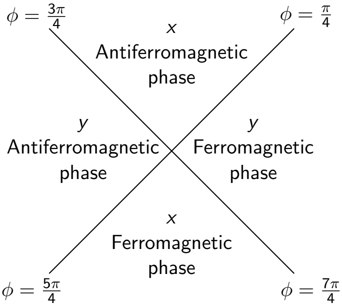
XY chain

$$H = -\cos\phi \sum_{j=1}^{N} \sigma_j^y \sigma_{j+1}^y + \sin\phi \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x \ , \quad N = \text{odd} \ ,$$





In the non-frustrated model:



Symmetries

$$\text{The parity operators:}\quad P_x = \prod_{j=1}^{N} \sigma_j^x \,, \quad P_y = \prod_{j=1}^{N} \sigma_j^y \,, \quad P_z = \prod_{j=1}^{N} \sigma_j^z$$

$$[H, P_\alpha] = 0 \,, \quad \alpha = x, y, z$$

$$\{P_\alpha, P_\beta\} = 0 \,, \quad \alpha \neq \beta$$

Consequences:

- Exact (at least) two-fold degeneracy of every eigenstate.
- Possible to break the parity symmetry in the GS of a finite system.
- Possible to have a non-zero magnetization in the GS of a finite system.

$$\langle \sigma_j^x \rangle_{\mathrm{GS}} \neq 0$$

- Hamiltonian has non-compatible symmetries: any GS breaks a symmetry of the Hamiltonian

The Magnetization

- The magnetization $\langle \sigma_j^x \rangle$ depends on the superposition coefficients.

$$|\mathrm{GS}\rangle = \alpha \, |\mathrm{GS+}\rangle + \beta \, |\mathrm{GS\text{-}}\rangle \ , \quad |\alpha|^2 + |\beta|^2 = 1$$

$$P_z \, |\mathrm{GS}\pm\rangle = \pm \, |\mathrm{GS}\pm\rangle$$

$$\langle \mathrm{GS}| \, \sigma_j^x \, |\mathrm{GS}\rangle = \alpha^* \beta \, \langle \mathrm{GS}+| \, \sigma_j^x \, |\mathrm{GS}-\rangle + \mathrm{c.c.}$$

- The order in the system is described by the maximal possible value.

In the thermodynamic limit of non-frustrated models:

$$| \, \langle \mathrm{GS}+| \, \sigma_j^x \, |\mathrm{GS}-\rangle \, | = \sqrt{\lim_{r \to \infty} \langle \sigma_j^x \sigma_{j+r}^x \rangle}$$

Computing the Magnetization

$$P_z \left|\mathrm{GS}+\right\rangle = \left|\mathrm{GS}+\right\rangle \ , \quad P_z \left|\mathrm{GS}-\right\rangle = -\left|\mathrm{GS}-\right\rangle$$

$$\{P_x, P_z\} = 0$$

$$P_x \left|\mathrm{GS}+\right\rangle = \left|\mathrm{GS-}\right\rangle$$

$$\left\langle\mathrm{GS}+\right| \sigma_j^x \left|\mathrm{GS}-\right\rangle = \left\langle\mathrm{GS}+\right| \sigma_j^x P_x \left|\mathrm{GS}+\right\rangle = \left\langle\mathrm{GS}+\right| \prod_{l\neq j}^{N} \sigma_l^x \left|\mathrm{GS}+\right\rangle$$

The Exact Solution

spins $\xrightarrow{\text{Jordan-Wigner transformation}}$ fermions

decomposition in two parity sectors ($P_z = +1$ and $P_z = -1$)

$$H = \frac{1+P_z}{2} H^+ \frac{1+P_z}{2} + \frac{1-P_z}{2} H^- \frac{1-P_z}{2}$$

$\xrightarrow{\text{Bogoliubov transformation}}$ free fermions in each sector

$$H^+ = (-\cos\phi + \sin\phi)\left(\chi_0^\dagger \chi_0 - \frac{1}{2}\right) + \sum_{q\in\{1,2,\ldots,N-1\}} \Lambda_q \left(\chi_q^\dagger \chi_q - \frac{1}{2}\right)$$

$$H^- = (\cos\phi - \sin\phi)\left(\chi_{\frac{N}{2}}^\dagger \chi_{\frac{N}{2}} - \frac{1}{2}\right) + \sum_{q\in\{\frac{1}{2},\frac{3}{2},\ldots,N-\frac{1}{2}\},q\neq\frac{N}{2}} \Lambda_q \left(\chi_q^\dagger \chi_q - \frac{1}{2}\right)$$

$$\text{mode energies:} \quad \Lambda_q = [1 - \sin 2\phi \ \cos(4\pi q/N)]^{1/2}$$

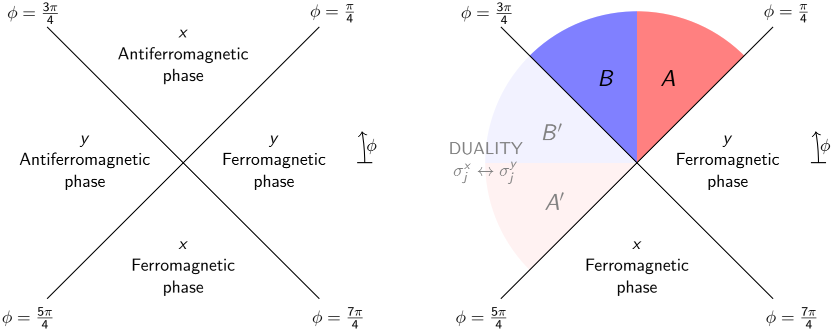
$$H = \frac{1+P_z}{2} H^+ \frac{1+P_z}{2} + \frac{1-P_z}{2} H^- \frac{1-P_z}{2}$$

Constructing the ground state: Adding excitations $\chi_q^\dagger$ to the states of the BCS form to satisfy the parity requirements.

Where antiferromagnetic interactions dominate:

- GS of H is not the GS of $H^{\pm}$.
- gapless, $\mathrm{gap} \sim \frac{1}{N^2}$

$$H = -\cos\phi \sum_{j=1}^{N} \sigma_j^y \sigma_{j+1}^y + \sin\phi \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x$$



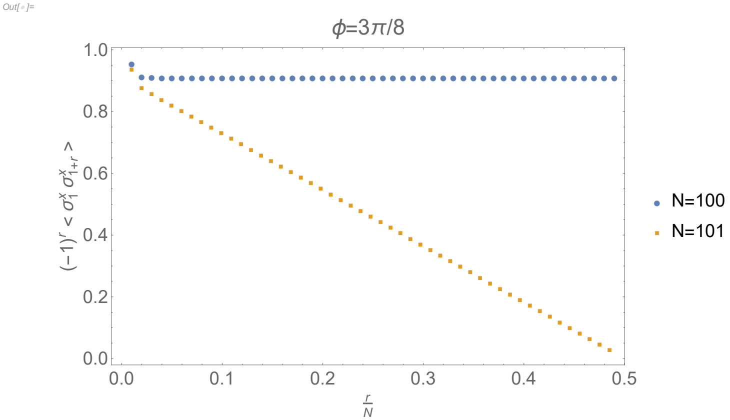
Non-frustrated model.

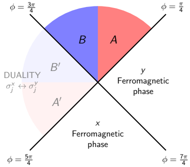
Frustrated model.

Region A: GS degeneracy $= 2$

Region B: GS degeneracy $= 4$, possible to break the translational invariance

The two-point correlator



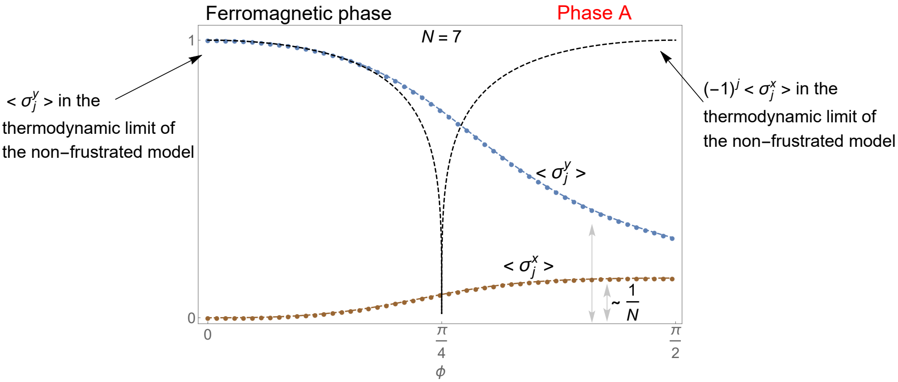


Regions A and B:

$$\langle \sigma_j^x \sigma_{j+r}^x \rangle = (-1)^r (1 - \cot^2 \phi)^{\frac{1}{2}} \left(1 - \frac{2r}{N} \right)$$

result analogous to [Dong et al J. Stat. Mech. 2016]

Phase A - Mesoscopic Magnetization



$$\text{Phase A: } \langle \sigma_j^x \rangle = \frac{1}{N}(1 - \cot^2 \phi)^{\frac{1}{4}}$$

Phase B - Breaking of Translational Invariance

$$|\mathrm{GS}\rangle = \alpha_1 |\mathrm{GS}_1+\rangle + \alpha_2 |\mathrm{GS}_2+\rangle + \beta_1 |\mathrm{GS}_1-\rangle + \beta_2 |\mathrm{GS}_2-\rangle$$

$|\mathrm{GS}_{1,2}\pm\rangle$ are eigenvectors of lattice translation operator with eigenvalues

$e^{i\left(\frac{\pi}{2}+\frac{\pi}{2N}\right)}, e^{-i\left(\frac{\pi}{2}+\frac{\pi}{2N}\right)}$

We can compute the magnetization in the case $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$

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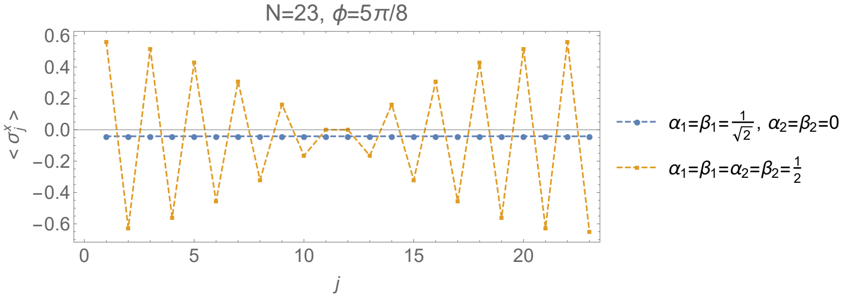

$\max_j \langle \sigma_j^x \rangle$ vs N

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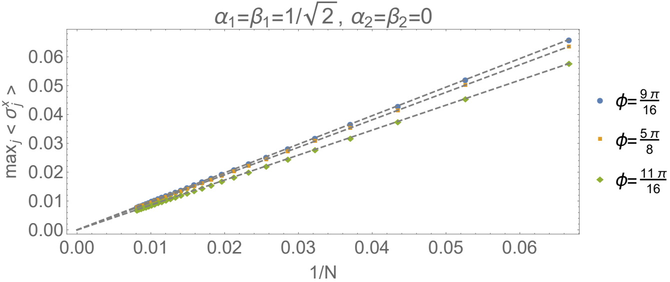

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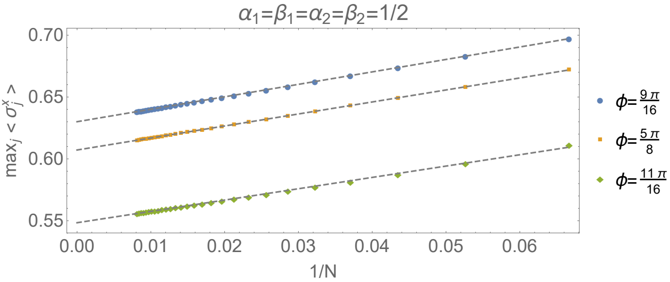
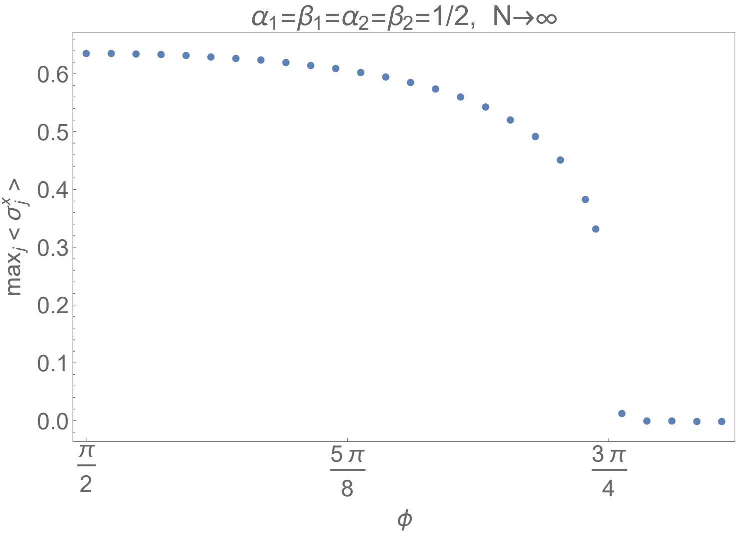
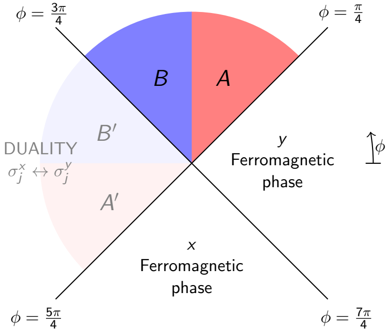


$\max_j \langle \sigma_j^x \rangle$ in the thermodynamic limit



Magnetization - Summary



Frustrated model.

Ferromagnetic phase (y)

GS degeneracy=2,

$$\langle \sigma_j^x \rangle = 0,\ \langle \sigma_j^y \rangle = (1 - \cot^2 \phi)^{\frac{1}{4}}$$

Phase A

GS degeneracy=2,

$$\langle \sigma_j^x \rangle, \langle \sigma_j^y \rangle \sim \frac{1}{N}$$

Phase B

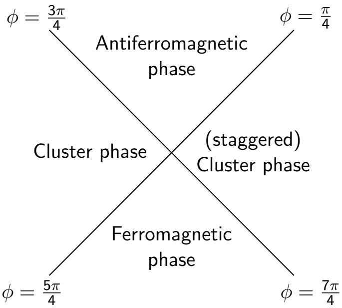
GS degeneracy=4, possible to break the translational invariance

$$\max_j \langle \sigma_j^x \rangle \sim a + \frac{b}{N},\ \max_j \langle \sigma_j^y \rangle \sim \frac{1}{N}$$

Cluster Ising Model

[Smacchia et. al. Phys.Rev.A 2011]

$$H = -\cos\phi \sum_{j=1}^{N} \sigma_{j-1}^{y} \sigma_{j}^{z} \sigma_{j+1}^{y} + \sin\phi \sum_{j=1}^{N} \sigma_{j}^{x} \sigma_{j+1}^{x}$$



String operator:

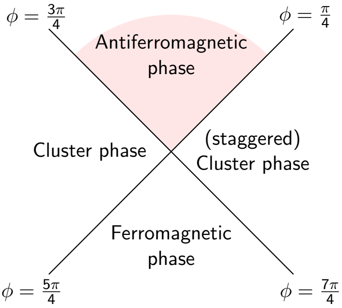
$$O_s(r) = \sigma_1^y \sigma_2^x \left(\prod_{l=3}^{r} \sigma_l^z \right) \sigma_{r+1}^x \sigma_{r+2}^y$$

Cluster phase:

$$\lim_{r\to\infty} \langle O_s(r) \rangle = (1 - \tan^2\phi)^{\frac{3}{4}}$$

Results

$N = 2M + 1$, Periodic Boundary Conditions



The cluster phase cannot be frustrated.

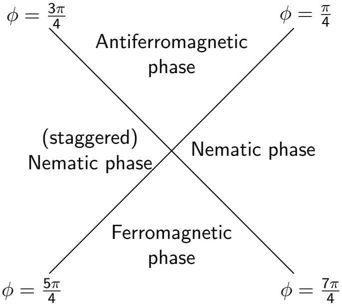
In AF phase of a finite system:

- GS is 3-fold degenerate when N is divisible by 3.
- Breaking of translational symmetry, but not parity symmetry.

2-Cluster Ising Model

[Giampaolo et. al. Phys.Rev.A 2015]

$$H = -\cos\phi \sum_{j=1}^{N} \sigma_{j-1}^{y}\sigma_{j}^{z}\sigma_{j+1}^{z}\sigma_{j+2}^{y} + \sin\phi \sum_{j=1}^{N} \sigma_{j}^{x}\sigma_{j+1}^{x}$$



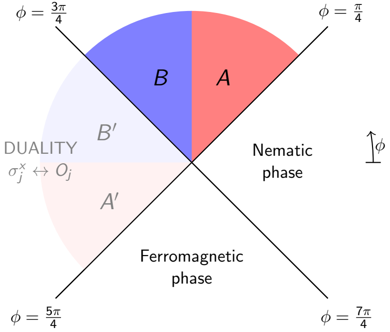
Nematic operator: $O_j = \sigma_{j-1}^{y}\sigma_{j}^{x}\sigma_{j+1}^{y}$

Nematic phase:

$$\lim_{r\to\infty} \langle O_j O_{j+r} \rangle = 1 - \tan^2\phi$$

- The same symmetries as in the XY chain.
- Duality: $\phi \leftrightarrow \frac{3\pi}{2} - \phi$, $O_j \leftrightarrow \sigma_j^x$

$$H = -\cos\phi \sum_{j=1}^{N} \sigma_{j-1}^{y}\sigma_{j}^{z}\sigma_{j+1}^{z}\sigma_{j+2}^{y} + \sin\phi \sum_{j=1}^{N} \sigma_{j}^{x}\sigma_{j+1}^{x}$$



Frustrated model.

The nematic operator O_j plays the role of σ_j^y in the XY chain.

The results are qualitatively the same.

Conclusion

- Different boundary conditions may result in a different behavior of the magnetization in the thermodynamic limit.
- Method of computing the magnetization in the XY chain.
- Three different phases characterized by the magnetization.
- Connection between frustration and supersymmetries?

Papers in preparation:

- Magnetization in the Frustrated XY chain
- Frustration and Nematic Order
- Frustration and Cluster Ising Model

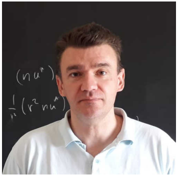
Possible future directions:

- dynamics of the models
- frustration and SUSY

People Involved

Fabio Franchini

Salvatore Marco Giampaolo



Domagoj Kuić