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Recent advances in T/U-dualities and generalized geometries
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Physical message/motivation of the talk:

Noncommutative (e.g. quantization deformed finite dimensional) phase space backgrounds have additional bi-/Hopf algebroid structure which allows considering deformations as coming from a generalization of Drinfeld twists.

A variant of the formalism can also be applied to Lie algebroids, e.g. at target spaces of sigma models. In principle, the formalism could be modified for deformations of phase spaces of classical *field* theories. One can tensor modules over bialgebroids hence the "symmetry" passes to bundles, differential calculi, multiparticle states, Fock space etc.

Classical Drinfeld twist

$\mathcal{F} \in H \otimes H$ (or $\mathcal{F}^{-1}$ in the usual convention) is a Drinfeld twist for a bialgebra $(H, \mu, \Delta : H \to H \otimes H, \epsilon : H \to \mathbb{C})$ if the 2-cocycle condition

$$(\Delta \otimes \mathrm{id})(\mathcal{F})(\mathcal{F} \otimes 1) = (\mathrm{id} \otimes \Delta)(\mathcal{F})(1 \otimes \mathcal{F}) \tag{1}$$

and the counitality $(\epsilon \otimes \mathrm{id})(\mathcal{F}) = 1_{\mathrm{H}} = (\mathrm{id} \otimes \epsilon)(\mathcal{F})$ hold.

Then in any H-module algebra $(\mathrm{A}, \mu_{\mathrm{A}})$ one has a deformed product $\mathrm{a} \star \mathrm{b} = \mu_{\mathrm{A}}\mathcal{F}(\mathrm{a} \otimes \mathrm{b})$ and coproduct $\Delta(\mathrm{h}) = \mathcal{F}^{-1}\Delta(\mathrm{h})\mathcal{F}$

Some candidates of Drinfeld twists have $\mathcal{F} \in \mathrm{B} \otimes \mathrm{B}$ where B is bigger *ambient Hopf algebra* than H at hand. Aschieri et al. allow for universal enveloping of Lie algebra of vector fields $\mathrm{B} = \mathrm{U}(\mathrm{VectM})$. Physically it should be $\mathrm{Diff}(\mathrm{M})$ but not a Hopf algebra. E.g.

$$\mathrm{x}\frac{\mathrm{d}}{\mathrm{dx}} \cdot \mathrm{x}\frac{\mathrm{d}}{\mathrm{dx}} - \mathrm{x}\frac{\mathrm{d}}{\mathrm{dx}} = \mathrm{x}^2\frac{\mathrm{d}}{\mathrm{dx}} \cdot \frac{\mathrm{d}}{\mathrm{dx}}$$

NOT true in $\mathrm{U}(\mathrm{Vect}(\mathbb{R}))$ – but true in $\mathrm{Diff}(\mathrm{M})$ (anticipate also: Hopf algebr*oid*, universal enveloping of tangent bundle as Lie algebroid).

Hopf algebroids – an analogue of groupoids in noncommutative geometry, a formalism of quantum groupoids. Main examples coming from usual groupoids, weak Hopf algebras (quasiHopf weak Hopf algebras Mack, Schomerus 1992; weak bialgebras G. Böhm 1997) and now also, what is our focus, from noncommutative phase spaces as a version of Lu's quantum **action groupoids**.

For simplicity we shall not discuss antipode of the Hopf algebroid; then we discuss an associative bialgebroid. Main ingredients of a **bialgebroid**: **total algebra** H, **base algebra** A, and

- structure of a $\mathrm{A} \otimes \mathrm{A}^{\mathrm{op}}$-**ring** $\eta : \mathrm{A} \otimes \mathrm{A}^{\mathrm{op}} \to \mathrm{H}$; then left leg $\alpha := \eta(- \otimes 1_{\mathrm{A}^{\mathrm{op}}}) : \mathrm{A} \to \mathrm{H}$ and right leg $\beta := \eta(1_{\mathrm{A}} \otimes -) : \mathrm{A}^{\mathrm{op}} \to \mathrm{H}$, also called the source and target maps respectively; $\alpha(\mathrm{a})\beta(\mathrm{b}) = \eta(\mathrm{a} \otimes \mathrm{b}) = \beta(\mathrm{b})\alpha(\mathrm{a})$.
- H is A-bimodule via

$$\mathrm{a.h.b} := \alpha(\mathrm{a})\beta(\mathrm{b})\mathrm{h}$$

counital **coproduct** $\Delta : \mathrm{H} \to \mathrm{H} \otimes_{\mathrm{A}} \mathrm{H}$ (comonoid in H-bimodules) and some compatibilities

Sweedler notation

To state the properties recall a bit notation on Hopf algebras
Sweedler notation: $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$.

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$

$$\sum \sum a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)} = \sum \sum a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)}$$

so we write simply

$$\sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$$

“only the order matters”

Hopf algebra of functions

Hopf algebroid of a groupoid $(G_1, G_0, s, t : G_1 \to G_0, \circ : G_1 {}_t\times_s G_1 \to G_1, i : G_1 \to G_1)$ – an analogue of Hopf algebra of functions on a group

A **Hopf algebra** is a bialgebra $(B, m, \eta, \Delta, \epsilon)$ with an antipode map $S : B \to B^{op}$,

$$m \circ (S \otimes id) \circ \Delta = \eta \circ \epsilon = m \circ (id \otimes S) \circ \Delta.$$

For a group G, $Fun(G)$ is a Hopf algebra with $\Delta(f)(g_1, g_2) = f(g_1 \cdot g_2)$ (using $Fun(G \times G) \cong Fun(G) \otimes Fun(G)$ and the antipode $S : Fun(G)^{op} \to Fun(G)$ is $(Sf)(g) = f(g^{-1})$, $g \in G$, $f \in Fun(G)$

(Co)modules

Algebras have actions, **modules**: $\nu : A \otimes M \to M$.
Coalgebras have coactions, **comodules**: $\rho : M \to C \otimes M$.
Extend Sweedler to $\rho(m) = \sum m_{(-1)} \otimes m_{(0)}$.
Modules over bialgebras have a **tensor product**:
$\nu(a, m \otimes n) = \sum \nu_M(a_{(1)}, m) \otimes \nu_N(a_{(2)}, n)$; dually comodules over bialgebras have a tensor product. Over Hopf algebras we also have duals (via antipode).
In physics, comultiplication so that the Hilbert space of **multiparticle state** inherits symmetry via tensor product of representations and quantum numbers appropriately “add”.

An $A \otimes A^{op}$-ring (H, μ, α, β) and an A-coalgebra $(H, \Delta : H \to H \otimes_A H, \epsilon : H \to A)$ on an A-bimodule H form a **left associative A-bialgebroid** $(H, \mu, \alpha, \beta, \Delta, \epsilon)$ if they satisfy the following compatibility conditions:

(C1) the underlying A-bimodule structure of the A-coring structure is determined by the source map and target map (part of the $A \otimes A^{op}$-ring structure): $r.a.r' = \alpha(r)\beta(r')a$.

(C2) formula $\sum_\lambda h_\lambda \otimes f_\lambda \mapsto \epsilon(\sum_\lambda h_\lambda \alpha(f_\lambda))$ defines an action $H \otimes A \to A$ which extends the left regular action $A \otimes A \to A$ along the inclusion $A \otimes A \xrightarrow{\alpha \otimes A} H \otimes A$.

(C3) the linear map $h \otimes (g \otimes k) \mapsto \Delta(h)(g \otimes k)$, $H \otimes (H \otimes H) \to H \otimes H$, induces a well defined action $H \otimes (H \otimes_A H) \to H \otimes_A H$.

The condition (C1) implies that the kernel $\mathrm{I}_{\mathrm{A}} = \mathrm{Ker}\,\pi$ of the projection map

$$\pi : \mathrm{H} \otimes_{\mathbb{R}} \mathrm{H} \to \mathrm{H} \otimes_{\mathrm{A}} \mathrm{H}$$

of H-bimodules is a **right ideal** in the algebra $\mathrm{H} \otimes_{\mathbb{R}} \mathrm{H}$, generated by the set of elements of the form $\beta(\mathrm{a}) \otimes 1 - 1 \otimes \alpha(\mathrm{a})$:

$$\mathrm{I}_{\mathrm{A}} = \{\, \beta(\mathrm{a}) \otimes_{\mathbb{R}} 1 - 1 \otimes_{\mathbb{R}} \alpha(\mathrm{a}) \,|\, \mathrm{a} \in \mathrm{A} \,\} \cdot (\mathrm{H} \otimes_{\mathbb{R}} \mathrm{H}) \qquad (2)$$

The third condition (C3) is here stated in the form of Lu. Let

$$\mathrm{H} \times_{\mathrm{A}} \mathrm{H} = \Big\{ \sum \mathrm{b}_i \otimes \mathrm{b}'_i \in \mathrm{H} \otimes_{\mathrm{A}} \mathrm{H} \,\Big|\, \sum_i \mathrm{b}_i \otimes \mathrm{b}'_i \alpha(\mathrm{a}) = \sum_i \mathrm{b}_i \beta(\mathrm{a}) \otimes \mathrm{b}'_i,\ \forall \mathrm{a} \in \mathrm{A} \Big\}$$

which is an H-subbimodule of $\mathrm{H} \otimes_{\mathrm{A}} \mathrm{H}$. Then the **Takeuchi product** $\mathrm{H} \times_{\mathrm{A}} \mathrm{H}$ is, *unlike* $\mathrm{H} \otimes_{\mathrm{A}} \mathrm{H}$, an associative algebra with respect to the componentwise product.

Main example of a nc bialgebroid over commutative base is the Heisenberg-Weyl associative algebra as total space over the coordinate part as a base; more generally differential operators Diff(M) over $\mathrm{C}^\infty(\mathrm{M})$. Action groupoid for infinitesimal action of derivative part.
(Basic example of a noncommutative Hopf algebroid over a *commutative* base) $\mathrm{A} = \mathrm{C}^\infty(\mathrm{M})$ where M is a smooth manifold. $\mathrm{H} = \mathrm{Diff}(\mathrm{M})$ is the algebra of **differential operators** with smooth coefficients. Define $\Delta(\mathrm{D})(\mathrm{f}, \mathrm{g}) = \mathrm{D}(\mathrm{f} \cdot \mathrm{g})$. The base is commutative and $\alpha = \beta$ is the canonical embedding of functions into differential operators; the counit is taking the constant term. Here $\blacktriangleright$ denotes the usual action of differential operators on functions.

The tangent bundle TM is an example of a Lie algebroid; the universal enveloping of a Lie algebroid (more generally, Lie-Rinehart pair) is a Hopf algebroid over the commutative base. For the vector fields, $\mathrm{U}_{\mathrm{C}^\infty(\mathrm{M})}(\mathrm{Vect}(\mathrm{M})) \cong$ Diff(M), not true for universal enveloping as Lie algebras! Namely in Weyl algebra it is true and not true in the enveloping that

$$\mathrm{x}\frac{\mathrm{d}}{\mathrm{dx}} \cdot \mathrm{x}\frac{\mathrm{d}}{\mathrm{dx}} - \mathrm{x}\frac{\mathrm{d}}{\mathrm{dx}} = \mathrm{x}^2\frac{\mathrm{d}}{\mathrm{dx}} \cdot \frac{\mathrm{d}}{\mathrm{dx}}$$

Thus the symmetry observables should not be looked upon within the usual enveloping which is a Hopf algebra but in the Lie algebroid enveloping which is a Hopf algebroid. Thus some no-go theorems for Drinfeld twists are non-physical and we need to go beyond.

Drinfeld twist for bialgebroids

$\mathcal{F} \in \mathrm{H} \otimes_{\mathrm{A}} \mathrm{H}$ is a Drinfeld twist for a left A-bialgebroid $(\mathrm{H}, \mu, \alpha, \beta, \Delta, \epsilon)$ if the 2-cocycle condition

$$(\Delta \otimes_{\mathrm{A}} \mathrm{id})(\mathcal{F})(\mathcal{F} \otimes_{\mathrm{A}} 1) = (\mathrm{id} \otimes_{\mathrm{A}} \Delta)(\mathcal{F})(1 \otimes_{\mathrm{A}} \mathcal{F}) \tag{3}$$

and the counitality $(\epsilon \otimes_{\mathrm{A}} \mathrm{id})(\mathcal{F}) = 1_{\mathrm{H}} = (\mathrm{id} \otimes_{\mathrm{A}} \epsilon)(\mathcal{F})$ hold.

(inverse cocycle) In terms of $\mathcal{F}^{-1}$ and A_* (next page), we can alternatively write the cocycle condition (3) as

$$(\mathcal{F}^{-1} \otimes_{\mathrm{A}_\star} 1)(\Delta \otimes_{\mathrm{A}_\star} \mathrm{id})(\mathcal{F}^{-1}) = (1 \otimes_{\mathrm{A}_\star} \mathcal{F}^{-1})(\mathrm{id} \otimes_{\mathrm{A}_\star} \Delta)(\mathcal{F}^{-1}).$$

If H is a left A-bialgebroid then (with notation $\mathcal{F} = \mathrm{f}^1 \otimes \mathrm{f}_1$) the formula

$$\mathrm{a} \star \mathrm{b} = \mu\mathcal{F}(\blacktriangleright \otimes \blacktriangleright)(\mathrm{f} \otimes \mathrm{g}) = (\mathrm{f}^1 \blacktriangleright \mathrm{a})(\mathrm{f}_1 \blacktriangleright \mathrm{b}) \tag{4}$$

defines an associative algebra $\mathrm{A}_\star = (\mathrm{A}, \star)$ structure on A with the same unit; the formulas $\alpha_{\mathcal{F}}(\mathrm{a}) = \alpha(\mathrm{f}^1 \blacktriangleright \mathrm{a})\mathrm{f}_1$ and $\beta_{\mathcal{F}}(\mathrm{a}) = \beta(\mathrm{f}^1 \blacktriangleright \mathrm{a})\mathrm{f}_1$ define respectively an algebra homomorphism and antihomomorphism $\mathrm{A}_\star \to \mathrm{H}$ turning H into an $\mathrm{A}_\star$-ring; the formula

$$\Delta_{\mathcal{F}}(\mathrm{h}) = \mathcal{F}^{-1}\Delta(\mathrm{h})\mathcal{F}$$

defines a map $\Delta_{\mathcal{F}} : \mathrm{H} \to \mathrm{H} \otimes_{\mathrm{A}_\star} \mathrm{H}$ which is coassociative and counital with the same counit. Moreover, $\mathrm{H}_{\mathcal{F}} = (\mathrm{H}, \mu, \alpha_{\mathcal{F}}, \beta_{\mathcal{F}}, \Delta_{\mathcal{F}}, \epsilon)$ is a left $\mathrm{A}_\star$-bialgebroid.

deformation quantization

Ping Xu (2000) extends the base algebra $\mathrm{C}^\infty(\mathrm{M})$ in $\mathrm{Diff}(\mathrm{M})$ to $\mathrm{C}^\infty(\mathrm{M})[[\mathrm{h}]]$ in $\mathrm{Diff}(\mathrm{M})$.

Ping Xu's theorem. If $(\mathrm{M}, \{,\})$ is Poisson manifold and the formal bidifferential operator $\mathcal{F} \in \mathcal{D}[[\mathrm{h}]]$ defines a deformation quantization of $(\mathrm{M}, \{,\})$ with $\mathrm{a} \star \mathrm{b} = \mu\mathcal{F}(\mathrm{f} \otimes \mathrm{g})$. Then $\mathcal{F}$ is a Drinfeld twist for the left $\mathrm{C}^\infty(\mathrm{M})[[\mathrm{h}]]$-bialgebroid of formal power series in regular differential operators Diff$[[\mathrm{h}]]$. Consequently, each deformation quantization defines also a deformation of that bialgebroid.

For Lie type noncommutativity work with Meljanac; later M. Stojić.
$\mathfrak{g}$ – fixed Lie algebra over $\mathbb{R}$ with basis $\hat{x}_1, \ldots, \hat{x}_n$
$U(\mathfrak{g})$ – universal enveloping algebra of $\mathfrak{g}$
$S(\mathfrak{g})$ – symmetric algebra of $\mathfrak{g}$.
$\hat{x}_1, \ldots, \hat{x}_n$ generate $U(\mathfrak{g})$ (noncommutative coordinates)
$x_1, \ldots, x_n$ generate $S(\mathfrak{g})$ (commutative coordinates)

$$[\hat{x}_\mu, \hat{x}_\nu] = C^\lambda_{\mu\nu} \hat{x}_\lambda. \quad (5)$$

$\partial^1, \ldots, \partial^n$ – algebraically dual basis of $\mathfrak{g}^*$, proportional to momenta $p^1, \ldots, p^n$
$\hat{S}(\mathfrak{g}^*)$ – formal completion of $S(\mathfrak{g}^*)$

Define also

$$\mathcal{O} := \exp(\mathcal{C}) \in \mathrm{M_n}(\hat{\mathrm{S}}(\mathfrak{g}^*))$$

$$\phi := \frac{-\mathcal{C}}{\mathrm{e}^{-\mathcal{C}} - 1} = \sum_{\mathrm{N}=0}^{\infty} \frac{(-1)^{\mathrm{N}}\mathrm{B_N}}{\mathrm{N}!}\mathcal{C}^{\mathrm{N}}, \qquad \tilde{\phi} := \frac{\mathcal{C}}{\mathrm{e}^{\mathcal{C}} - 1} = \sum_{\mathrm{N}=0}^{\infty} \frac{\mathrm{B_N}}{\mathrm{N}!}\mathcal{C}^{\mathrm{N}}, \tag{6}$$

where $\mathrm{B_N}$ are the Bernoulli numbers.

$$\mathcal{C}^{\alpha}_{\beta} := \mathrm{C}^{\alpha}_{\beta\gamma}\partial^{\gamma} \in \hat{\mathrm{S}}(\mathfrak{g}^*), \tag{7}$$

$$\hat{\mathrm{x}}^{\phi}_{\rho} := \sum_{\tau} \mathrm{x}_{\tau}\phi^{\tau}_{\rho}, \quad \hat{\mathrm{y}}^{\phi}_{\rho} := \sum_{\tau} \mathrm{x}_{\tau}\tilde{\phi}^{\tau}_{\rho} = \beta(\hat{\mathrm{x}}^{\phi}).$$

$$\hat{y}^\phi_\nu = \hat{x}^\phi_\mu \mathcal{O}^\mu_\nu,$$

$$[\hat{x}^\phi_\mu, \hat{y}^\phi_\nu] = 0.$$

Note

$$\phi : \mathrm{U}(\mathfrak{g}) \to \mathrm{End}^{\mathrm{op}}(\hat{\mathrm{S}}(\mathfrak{g}^*)).$$

$$\phi^\mu_\nu = \phi(\partial^\mu)(\hat{x}_\nu)$$

Hence we can form a smash product algebra $\mathrm{U}(\mathfrak{g})\sharp\hat{\mathrm{S}}(\mathfrak{g}^*)$ – in fact Heisenberg double – contains both $\hat{x}_\mu$ and p^ν and $\hat{\mathrm{S}}(\mathfrak{g}^*) \cong \mathrm{U}(\mathfrak{g})^*$ dual Hopf algebra! $\mathrm{A}^{-1} = (\phi^\mu_\nu)$ inverse Maurer-Cartan $d\mathrm{A} + \mathrm{A} \wedge \mathrm{A} = 0$
Hopf algebroid: $\Delta(\hat{x}_\mu) = \hat{x}_\nu \otimes 1$, $\Delta(p^\mu)$ is from the dual Hopf algebra $\mathrm{U}(\mathfrak{g}^*)$. This is a case of Lu/Brzeziński-Militaru scalar extension by bc Yetter-Drinfeld module algebra. $\hat{y}_\mu \blacktriangleright u = u\hat{x}_\mu$.

In symmetric ordering, the deformed coproduct Δ on $\hat{\mathrm{S}}(\mathfrak{g}^*) \cong \mathrm{U}(\mathfrak{g})^*$ is given by

$$\Delta\partial^\mu = 1 \otimes \partial^\mu + \partial^\alpha \otimes [\partial^\mu, \hat{\mathrm{x}}_\alpha] + \frac{1}{2}\partial^\alpha\partial^\beta \otimes [[\partial^\mu, \hat{\mathrm{x}}_\alpha], \hat{\mathrm{x}}_\beta] + \ldots$$

or

$$\Delta\partial^\mu = \exp(\partial^\alpha \otimes \mathrm{ad}\,(-\hat{\mathrm{x}}_\alpha))(1 \otimes \partial^\mu) = \exp(\mathrm{ad}(-\partial^\alpha \otimes \hat{\mathrm{x}}_\alpha))(1 \otimes \partial^\mu).$$

$$\Delta(\hat{\mathrm{p}}^\mu) = \mathcal{F}_{\mathrm{L}}^{-1}\Delta_0(\hat{\mathrm{p}}^\mu)\mathcal{F}_{\mathrm{L}} \tag{8}$$

where $\mathcal{F}_{\mathrm{L}}$ is the product of the two exponentials:

$$\mathcal{F}_{\mathrm{L}} = \exp(-\partial^\rho \otimes \mathrm{x}_\rho)\exp(\partial^\sigma \otimes \hat{\mathrm{x}}_\sigma) \tag{9}$$

$\mathbb{R}$-linear map $\psi : \mathfrak{g} \to \mathfrak{g}$ **automorphism** of Lie algebra $\mathfrak{g}$ if $[\psi(x), \psi(y)] = \psi([x, y])$ what for $\psi(\hat{x}_\alpha) = \hat{x}_\beta M^\beta_\alpha$ takes the form

$$[\hat{x}_\alpha M^\alpha_\mu, \hat{x}_\beta M^\beta_\nu] = C^\gamma_{\alpha\beta} \hat{x}_\gamma M^\alpha_\mu M^\beta_\nu,$$

degenerate Hopf **pairing** $\mathrm{Fun}(\mathrm{Aut}\mathfrak{g}) \otimes U(\mathfrak{g}) \to \mathbb{R}$ by

$$\langle f, X_1 X_2 \ldots X_r \rangle =$$

$$\frac{d}{dt_n}\frac{d}{dt_{n-1}} \cdots \frac{d}{dt_1} f(\exp(t_1 \,\mathrm{ad}\, X_n) \cdots \exp(t_n \,\mathrm{ad}\, X_1))|_{t_1=0,\ldots,t_n=0}$$

At $\mathfrak{g}$, this can be understood as $(\mathrm{ad}X)(f)$ as $\mathrm{ad}X$ is a tangent vector at $\mathrm{Inn}\,\mathfrak{g} \subset \mathrm{Aut}\,\mathfrak{g}$.

$$C^{\sigma}_{\mu\nu} M^{\gamma}_{\sigma} = C^{\gamma}_{\alpha\beta} M^{\alpha}_{\mu} M^{\beta}_{\nu} \tag{10}$$

hence $\mathrm{Aut}\mathfrak{g}$ can be identified with the affine algebraic subgroup of the automorphism of the underlying vector subspace.
The same equation as for $\mathcal{O}$!!!

The degenerate pairing induces a left **Hopf** action $\blacktriangleright$: $\mathrm{Fun}(\mathrm{Aut}(\mathfrak{g})) \otimes \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})$

$$\mathrm{f} \blacktriangleright \mathrm{D} = \mathrm{D}_{(1)} \langle \mathrm{f}, \mathrm{D}_{(2)} \rangle$$

(Stojić, ZŠ) Algebra U satisfy the compatibilities with the left H-action $\blacktriangleright$ and the right H-coaction $\rho : \hat{\mathrm{x}}_\mu \mapsto \hat{\mathrm{x}}_\nu \otimes \mathrm{M}^\nu_\mu$ (giving $\beta : \hat{\mathrm{x}}_\mu \mapsto \hat{\mathrm{x}}_\nu \sharp \mathrm{M}^\nu_\mu$!!!) making U into a braided commutative Yetter-Drinfeld module algebra.

YD condition where $\rho(X) = X_{[0]} \otimes X_{[1]}$, $\Delta(f) = f_{(1)} \otimes f_{(2)}$.

$$f_{(1)} X_{[0]} \otimes f_{(2)} X_{[1]} = f_{(1)} \blacktriangleright X_{[0]} \otimes f_{(2)} X_{[1]}$$

Braided commutativity:

$$X_{[0]}(X_{[1]} \blacktriangleright Y) = YX$$

For groups and groupoids YD condition is just equivariance for adjoint action!
YD modules are modules over DRINFELD double. Heisenberg double an example and dual Hopf algebra an example. bc YD module algebra A gives a Hopf A-algebroid (Lu, Brzezinski-Militaru) $\mathrm{H}\sharp\mathrm{A}$, $\Delta(\mathrm{h}) = \mathrm{h}_{(1)} \otimes \mathrm{h}_{(2)}$.
Vafa, Dixon, Jeffreys 1985 Strings on orbifolds: twisted and multisectors and orbifold Euler characteristics. Lupercio, Uribe, Ruan around 2000: twisted sectors correspond to work over loop orbifold which corresponds to the inertia groupoid – selfmorphisms as objects and conjugations as morphisms. Hinich and independently Škoda 2003/2004: Drinfeld-Majid center of the monoidal category of bundles over an orbifold is equivalent to monoidal category of equivariant bundles over inertia orbifold. Baranovsky: orbifold cohomology from cyclic homology. antiYD modules coefficients for Hopf cyclic homology (Rangipour et al).

Kowalzig: YD module algebras over Hopf algebroids give Batalin-Vilkovisky modules over Gerstenhaber algebra (graded Poisson) (noncommutative differential calculus).
Graded Poisson: cup product and graded Poisson bracket (on $V[1]$)
BV module: graded module (think of forms) formal Lie derivative $\mathcal{L}$ with mixed Leibniz rule and differential B satisfying Cartan homotopy formula $\mathcal{L}_\phi(x) = B(\phi \cap x) - (-1)^p \phi \cap B(x)$
Q: what it gives in our case ? (open)

$\mathcal{M}$ – manifold. A **Lie algebroid** over M a smooth vector bundle $\mathrm{A} \to \mathcal{M}$ with

- a $\mathbb{k}$-Lie bracket $[,]$ on the space of sections of A;
- a map of vector bundles $\mathrm{a} : \mathrm{A} \to \mathrm{T}\mathcal{M}$, called the **anchor map**, such that

$$[\mathrm{X}, \mathrm{fY}] = \mathrm{f}[\mathrm{X}, \mathrm{Y}] + \mathrm{a}(\mathrm{X})(\mathrm{f})\mathrm{Y}$$

for all sections X, Y of A, $\mathrm{f} \in \mathrm{C}^\infty(\mathcal{M})$.

More generally, let

- $\mathcal{O}$ – a commutative algebra over $\mathbb{R}$
- L – a symmetric $\mathcal{O}$-bimodule

One says that $\mathcal{O}$ is a **Lie-Rinehart algebra** if there is a $\mathbb{R}$-linear Lie bracket $[,]$ on L, and a morphism of $\mathcal{O}$-modules $\mathrm{a} : \mathrm{L} \to \mathrm{Der}_{\mathbb{R}}(\mathcal{O})$, such that

$$[\mathrm{X}, \mathrm{fY}] = \mathrm{f}[\mathrm{X}, \mathrm{Y}] + \mathrm{a}(\mathrm{X})(\mathrm{f})\mathrm{Y}. \tag{11}$$

$\mathrm{Der}_{\mathbb{R}}(\mathcal{O})$ Lie algebra of $\mathbb{R}$-linear derivations of $\mathcal{O}$.

The universal enveloping algebra $\mathrm{U}(\mathrm{L})$ of a Lie algebroid or Lie-Rinehart algebra is the tensor algebra $\mathrm{T}_{\mathcal{O}}\mathrm{L}$ over module $\mathcal{O}$ modulo the ideal, generated by the ideal of the relations

$$\mathrm{XY} - \mathrm{YX} = [\mathrm{X}, \mathrm{Y}], \tag{12}$$

$$\mathrm{XfY} - \mathrm{fXY} = \mathrm{a}(\mathrm{X})(\mathrm{f})\mathrm{Y}. \tag{13}$$

We want to repeat the construction of the Heisenberg double Hopf algebroid $\mathrm{Fun}(\mathrm{Aut}\,\mathfrak{g})\sharp\mathrm{U}(\mathfrak{g})$ with a Lie algebra $\mathfrak{g}$ replaced by a Lie algebroid or even a Lie-Rinehart algebra L.

$\mathrm{U}(\mathrm{L})$ has a comultiplication over $\mathcal{O}$ inherited from $\mathrm{T}_{\mathcal{O}}\mathrm{L}$ There is a PBW map from the symmetric algebra over $\mathcal{O}$ to the universal enveloping but is not respecting the coproduct! In presence of a Lie algebroid connection it can be corrected. Like in Lie algebra case.

Choose a basis e_α of $\Gamma\mathrm{L}$ as a $\mathrm{C}^\infty(\mathcal{M})$-module. The automorphism of L as a Lie algebroid is an automorphism as a vector bundle, given by a matrix M with entries in $\mathrm{C}^\infty(\mathcal{M})$ such that it commutes with the anchor map a and preserves the bracket. In terms of M,

$$\mathrm{a}(\mathrm{e}_\alpha) = \mathrm{M}_\alpha^\beta \mathrm{a}(\mathrm{e}_\beta)$$

$$\mathrm{M}_\mu^\alpha \mathrm{M}_\nu^\beta \mathrm{C}_{\alpha\beta}^\rho - \mathrm{M}_\mu^\sigma \mathrm{a}(\mathrm{e}_\sigma)(\mathrm{M}_\nu^\rho) + \mathrm{M}_\nu^\sigma \mathrm{a}(\mathrm{e}_\sigma)(\mathrm{M}_\mu^\rho) = \mathrm{C}_{\alpha\beta}^\gamma \mathrm{M}_\gamma^\rho$$

These are algebraic conditions on M *and derivatives* of M in the setup of infinite-dimensional geometry over a *ring* $\mathrm{C}^\infty(\mathcal{M})$.

We do not know how to do such differential algebra. However, the matrix function G – if the duals of $\mathrm{U}(\mathrm{L})^*$ and $\hat{\mathrm{S}}_{\mathcal{O}}(\mathrm{L}^*)$ are properly identified via the transpose of the corrected coexponential map – has a meaning (passing between left and right invariant vector fields);

Therefore take the $\mathcal{O}$-subalgebra generated by these matrix elements. This enables constructions of structure maps (including the pairing) like in Hopf algebroid (but some maps exist only dually).

Szlachanyi's approach known to work: extension by YD module of commutative Hopf algebroid $\mathrm{U}(\mathrm{L})$ (over $\mathrm{C}^\infty(\mathrm{M})$).

Notice that canonical transformation of the exchange of momenta and coordinates *changes* the Hopf algebroid. Phase space Hopf algebroid knows which of the two is reflected in algebraic and which in coalgebraic sector. Only as algebra, the deformed and undeformed case are the same at the phase space level, but the configuration space part differs. So Hopf algebroid remembers the polarization and how much of the phase space Poisson noncommutativity is squeezed into the configuration part after the deformation quantization.

THANKS!