

# Extended Riemannian Geometry and Double Field Theory

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Based on:

- [arXiv:1611.02772](https://arxiv.org/abs/1611.02772) with Andreas Deser

**Aim: Clarify mathematical structures behind Double Field Theory**

## Double Field Theory (DFT)

- **target space formulations** useful (SUSY YM, SUGRA)
  - **T-duality** very interesting
- } **DFT**
- Tseytlin, Siegel, Hull, Zwiebach, ...
- Success, e.g.: **DFT action** on  $\mathbb{R}^{2D}$ :

$$S = \int d^{2D}x e^{-2d} \left( \frac{1}{8} \mathcal{H}_{MN} \partial^M \mathcal{H}_{KL} \partial^N \mathcal{H}^{KL} - \frac{1}{2} \mathcal{H}_{MN} \partial^M \mathcal{H}_{KL} \partial^L \mathcal{H}^{KN} - 2\partial^M d \partial^N \mathcal{H}_{MN} + 4\mathcal{H}_{MN} \partial^M d \partial^N d \right)$$

$\mathcal{H}_{MN}$ : generalized metric,  $d$ : DFT dilaton

- **Reproduces SUGRAs** previously not derived from string theory
- **sensible?** strange truncation of string modes? seems to work

Clarifying the mathematical structures underlying Double Field Theory.

- Massless string modes: metric  $g$ , 2-form  $B$ , dilaton  $\phi$ .
- $g$  and  $B$  transform **differently**, combine in **generalized metric**:

$$\mathcal{H}_{MN} = \begin{pmatrix} g_{\mu\nu} - B_{\mu\kappa}g^{\kappa\lambda}B_{\lambda\nu} & B_{\mu\kappa}g^{\kappa\nu} \\ -g^{\mu\kappa}B_{\kappa\nu} & g^{\mu\nu} \end{pmatrix} \quad \text{on } T^*M \oplus TM$$

- Manifest T-duality: **double space**, coords.:  $x^M = (x^\mu, x_\mu)$
- **Level matching**: Fields  $\phi$  in DFT satisfy  $\square\phi = 0$
- Algebra of fields: **strong section condition**:

$$\partial^M \phi \partial_M \psi = 0$$

- **Generalized Lie derivative** wrt.  $X = X^\mu \frac{\partial}{\partial x^\mu} + X_\mu \frac{\partial}{\partial x_\mu}$ :

$$\hat{\mathcal{L}}_X \mathcal{H}_{MN} = X^P \partial_P \mathcal{H}_{MN} + (\partial_M X^P - \partial^P X_M) \mathcal{H}_{PN} + (\partial_N X^P - \partial^P X_N) \mathcal{H}_{MP}$$

- $\hat{\mathcal{L}}_X$  form **Lie algebra**, but **not** representation of Lie algebra

$$\hat{\mathcal{L}}_X \hat{\mathcal{L}}_Y - \hat{\mathcal{L}}_Y \hat{\mathcal{L}}_X = \hat{\mathcal{L}}_{\mu_2(X,Y)},$$

- $\mu_2(X, Y)$  known as **C-bracket** (actual interpretation later)

Issues to address in a mathematical formulation:

- Issue 1: Algebraic structure behind **C-bracket**
- Issue 2: **Section condition is rather invidious:**

$$\partial^M \phi \partial_M \psi = 0$$

- Issue 3: **Section condition is often too strong**
- Issue 4: Reasonable version of **Doubled Riemannian Geometry**

$$\begin{aligned} \hat{\Gamma}_{MNK} = & -2(P\partial_M P)_{[NK]} - 2(\bar{P}_{[N}{}^P \bar{P}_{K]}{}^Q - P_{[N}{}^P P_{K]}{}^Q) \partial_P P_{QM} \\ & + \frac{4}{D-1} (P_{M[N} P_{K]}{}^Q + \bar{P}_{M[N} \bar{P}_{K]}{}^Q) (\partial_Q d + (P\partial^P P)_{[PQ]}) \end{aligned}$$

- Issue 5: **Global formulation of DFT**
- Issue 6: Same for **Exceptional field theory** (M-theory analogue)

- Massless string modes: metric  $g$ , 2-form  $B$ , dilaton  $\phi$ .
- Recall:  $B$ -field belongs to **gerbe/categorified principal bundle**
- Generalized Geometry: Courant/Lie 2-algebroid  $TM \oplus T^*M$
- Captures Lie 2-algebra of **symmetries** (gauge+diffeos)
- In particular: **Courant/Dorfman brackets**
- Picture allows for an **extension** to pre-Lie 2-algebroids
- Lie 2-algebra with **C-** and **D-**brackets
- Categorified Jacobi relations  $\Leftrightarrow$  **weakened section condition**
- Fully **algebraic** and **coordinate independent** picture
- Know how to **twist** pre-Lie 2-algebroids
- Potential **global picture** (work in progress)

## Part I : Generalized Geometry

- Gerbes lead to Courant algebroids
- Higher Lie algebroids/algebras as  $NQ$ -manifolds
- Symmetries from higher symplectic Lie algebroids

# Recall: Generalized Geometry

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Generalized Geometry provides a nice geometric description of the  $B$ -field.

**Ingredients:** massless string modes: metric  $g$ , 2-form  $B$ , dilaton  $\phi$ .

- Well-known:  $B$  belongs to connective structure of a **gerbe**  
Gawedzki 1987, Freed&Witten 1999

## Gerbes - Intuitive Picture

- Take your **favorite definition** of principal bundles
- Smooth Manifold  $\rightarrow$  **smooth categories**  $\mathcal{M}_1 \rightrightarrows \mathcal{M}_0$
- Smooth morphisms between manifolds  $\rightarrow$  **smooth functors**
- Structure group  $\rightarrow$  **special category with products**
- There are obvious categorified notions of:  
**fibration, surjection, diffeomorphism, free, transitive action**
- Result: **principal 2-bundles**
- Gerbe:** principal 2-bundle for  $U(1) \rightrightarrows *$

# Recall: Čech Description of a Gerbe

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Generalized Geometry provides a nice geometric description of the  $B$ -field.

## Cover/Surjective submersion

Manifold  $M$ , cover  $Y = \sqcup_i U_i \rightarrow M$ ,  $Y^{[2]} = \sqcup_{i,j} U_i \cap U_j, \dots$

## Principal $U(1)$ -bundle over $M$

$$U(1) \rightarrow P \rightarrow M, \quad g_{ij}g_{jk} = g_{ik}, \quad A_i = g_{ij}^{-1}(A_j + d)g_{ij}$$

Symmetries:  $U(1)$  gauge symmetry and diffeomorphisms.

## $U(1)$ -gerbe $\mathcal{G}$ over $M$

$$\begin{array}{ccccccc} U(1) & & \mathcal{G}_1 & & Y^{[2]} & & M \\ \Downarrow & \longrightarrow & \Downarrow & \longrightarrow & \Downarrow & \xrightarrow{\mathbb{R}} & \Downarrow \\ * & & \mathcal{G}_0 & & Y & & M \end{array}$$

$$h_{ijk}h_{ikl} = h_{ijl}h_{jkl}, \quad A_{ij} - A_{ik} + A_{jk} = d \log(h_{ijk}), \quad B_i - B_j = dA_{ij}$$

Categorified spaces  $\Rightarrow$  categorified symmetries!



Generalized Geometry provides a nice geometric description of the  $B$ -field.

Ordinary geometry: “Local” Atiyah Algebroid for  $U(1)$ -bundle

$$0 \rightarrow M \times i\mathbb{R} \xrightarrow{\hookrightarrow} TM \oplus i\mathbb{R} \xrightarrow{\text{pr}} TM \rightarrow 0$$

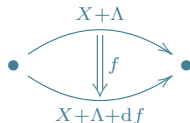
- Gauge potential  $A : TM \rightarrow i\mathbb{R}$  is defined by **section** of  $\text{pr}$
- Infinitesimal symm. (diffeos+gauge): sections of  $TM \oplus i\mathbb{R}$

Generalized Geometry: Courant algebroid for  $U(1)$ -gerbe

$$0 \rightarrow T^*M \xrightarrow{\hookrightarrow} TM \oplus T^*M \xrightarrow{\text{pr}} TM \rightarrow 0$$

- $g + B : TM \rightarrow T^*M$  again from **section** of  $\text{pr}$
- Infinitesimal symm. (diffeos+gauge): sections of  $TM \oplus T^*M$

not complete picture:

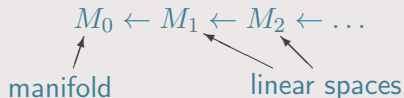


more later

NQ-manifolds, known from BRST quantization, provide very useful language.

## N-manifolds, NQ-manifold

- $\mathbb{N}_0$ -graded manifold with coordinates of degree  $0, 1, 2, \dots$



- **NQ-manifold**: vector field  $Q$  of degree 1,  $Q^2 = 0$
- **Physicists**: think ghost numbers, BRST charge, SFT

Examples:

- **Tangent algebroid**  $T[1]M$ ,  $\mathcal{C}^\infty(T[1]M) \cong \Omega^\bullet(M)$ ,  $Q = d$
- **Lie algebra**  $\mathfrak{g}[1]$ , coordinates  $\xi^a$  of degree 1:

$$Q = -\frac{1}{2} f_{ab}^c \xi^a \xi^b \frac{\partial}{\partial \xi^c} \quad , \quad \text{Jacobi identity} \Leftrightarrow Q^2 = 0$$

$NQ$ -manifolds provide an easy definition of  $L_\infty$ -algebras.

Lie  $n$ -algebroid:

$$M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \dots \leftarrow M_n \leftarrow * \leftarrow * \leftarrow \dots$$

Lie  $n$ -algebra or  $n$ -term  $L_\infty$ -algebra:

$$* \leftarrow M_1 \leftarrow M_2 \leftarrow \dots \leftarrow M_n \leftarrow * \leftarrow * \leftarrow \dots$$

Important example: Lie 2-algebra

- Graded vector space:  $W[1] \leftarrow V[2]$
- Coordinates:  $w^a$  of degree 1 on  $W[1]$ ,  $v^i$  of degree 2 on  $V[2]$
- Most general vector field  $Q$  of degree 1:

$$Q = -m_i^a v^i \frac{\partial}{\partial w^a} - \frac{1}{2} m_{ab}^c w^a w^b \frac{\partial}{\partial w^c} - m_{ai}^j w^a v^i \frac{\partial}{\partial v^j} - \frac{1}{3!} m_{abc}^i w^a w^b w^c \frac{\partial}{\partial v^i}$$

- Induces “brackets”/“higher products”:

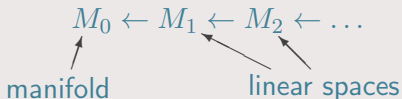
$$\mu_1(\tau_i) = m_i^a \tau_a, \quad \mu_2(\tau_a, \tau_b) = m_{ab}^c \tau_c, \quad \dots, \quad \mu_3(\tau_a, \tau_b, \tau_c) = m_{abc}^i \tau_i$$

- $Q^2 = 0 \Leftrightarrow$  Homotopy Jacobi identities, e.g.  $\mu_1(\mu_1(-)) = 0$
- Failure of Jacobi identity:  $\mu_2(x, \mu_2(y, z)) + \dots = \mu_1(\mu_3(x, y, z))$

Symplectic NQ-manifolds are very convenient to work with.

## Symplectic NQ-manifold

- $\mathbb{N}_0$ -graded manifold, vector field  $Q$  of degree 1,  $Q^2 = 0$



- **symplectic NQ-manifold**:  $\omega$  nondegenerate, closed,  $\mathcal{L}_Q \omega = 0$

Examples of symplectic NQ-manifolds:

- **Metric Lie algebra**:

$$(\mathfrak{g}[1], \omega = g_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta), \quad Q = -\frac{1}{2} f_{ab}^c \xi^a \xi^b \frac{\partial}{\partial \xi^c}$$

- **Symplectic manifolds** (symplectic Lie 0-algebroids):

$$(M, \omega), \quad Q = 0$$

- **Poisson manifolds** (symplectic Lie 1-algebroids):

$$T^*[1]M, \quad \omega = dp_\mu \wedge dx^\mu, \quad Q = \pi^{\mu\nu} p_\mu \frac{\partial}{\partial x^\nu}$$

- **Courant algebroids** (symplectic Lie 2-algebroids, next slide...)

The key geometric structure behind generalized geometry is locally  $T^*[2]T[1]M$ .

The symplectic NQ-manifold  $T^*[2]T[1]M$

**Local description**, choose  $M = \mathbb{R}^D$  as base manifold.

$$T^*[2]T[1]M = \mathbb{R}^D \oplus \mathbb{R}^D[1] \oplus \mathbb{R}^D[1] \oplus \mathbb{R}^D[2]$$
$$x^\mu \qquad \xi^\mu \qquad \zeta_\mu \qquad p_\mu$$

$$Q = \xi^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \zeta_\mu} \qquad \omega = dx^\mu \wedge dp_\mu + d\xi^\mu \wedge d\zeta_\mu$$

- functions on  $T^*[2]T[1]M$  of degree 0: **functions on  $M$**
- functions on  $T^*[2]T[1]M$  of degree 1: **sections of  $TM \oplus T^*M$**
- **Poisson bracket** induced by  $\omega$ , pairing on degree 1 functions:

$$\{X + \alpha, Y + \beta\} = \iota_X \alpha + \iota_Y \beta$$

# The $L_\infty$ -Algebra of a Symplectic $L_\infty$ -algebroid

Every symplectic  $L_\infty$ -algebroid comes with an  $L_\infty$ -algebra.

An underappreciated/widely unknown fact:

All symplectic Lie  $n$ -algebroids  $\mathcal{M}$  come with Lie  $n$ -algebra  $L(\mathcal{M})$ :

$$\begin{array}{ccccccc} C_0^\infty(\mathcal{M}) & \xrightarrow{Q} & C_1^\infty(\mathcal{M}) & \xrightarrow{Q} & \dots & \xrightarrow{Q} & C_{n-2}^\infty(\mathcal{M}) & \xrightarrow{Q} & C_{n-1}^\infty(\mathcal{M}) \\ L_{n-1}(\mathcal{M}) & \xrightarrow{Q} & L_{n-2}(\mathcal{M}) & \xrightarrow{Q} & \dots & \xrightarrow{Q} & L_1(\mathcal{M}) & \xrightarrow{Q} & L_0(\mathcal{M}) \end{array}$$

$$\mu_1(l) = \begin{cases} 0 & l \in C_{n-1}^\infty(\mathcal{M}) = L_0(\mathcal{M}) \\ Ql & \text{else} \end{cases}$$

$$\mu_2(l_1, l_2) = \frac{1}{2}(\{\delta l_1, l_2\} \pm \{\delta l_2, l_1\}), \quad \delta(l) = \begin{cases} Ql & l \in L_0(\mathcal{M}) \\ 0 & \text{else} \end{cases}$$

$$\mu_3(l_1, l_2, l_3) = -\frac{1}{12}(\{\{\delta l_1, l_2\}, l_3\} \pm \dots)$$

Roytenberg, Rogers, Fiorenza/Manetti, Getzler

- “Derived brackets” (Kosmann-Schwarzbach, Voronov, ...)
- Important class of examples next
- Cartan calculus ( $\Rightarrow$  Andreas’ talk)

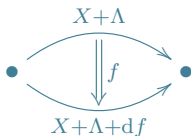
# Associated Algebraic Structures of Courant Algebroid

The key geometric structure behind generalized geometry is locally  $T^*[2]T[1]M$ .

- $\mathcal{M} = T^*[2]T[1]\mathbb{R}^D = \mathbb{R}^D \oplus \mathbb{R}^{2D}[1] \oplus \mathbb{R}^D[2]$ ,  $Q, \omega, \{-, -\}$
- Graded vector space:

$$\begin{aligned} \mathcal{C}_0^\infty(\mathcal{M}) \rightarrow \mathcal{C}_1^\infty(\mathcal{M}) &= \mathcal{C}^\infty(\mathbb{R}^D) \rightarrow \Gamma(T\mathbb{R}^D \oplus T^*\mathbb{R}^D) \\ &= \mathbb{L}_1(\mathcal{M}) \rightarrow \mathbb{L}_0(\mathcal{M}) \end{aligned}$$

- Derived brackets from differential  $\delta(\ell) = \begin{cases} Q\ell & \ell \in \mathbb{L}_0(\mathcal{M}) \\ 0 & \text{else} \end{cases}$
- Leibniz algebra from **Dorfman bracket**  $\nu_2(\ell_1, \ell_2) = \{\delta\ell_1, \ell_2\}$
- Lie 2-algebra from **Courant bracket**  $\mu_2(\ell_1, \ell_2) = \{\delta\ell_1, \ell_2\}$
- This is the **symmetry Lie 2-algebra** of corresponding gerbe!



The above picture is part of a larger story, involving GR coupled to  $n$ -form potentials.

More generally: Couple GR to  $n$ -form gauge potential

Vinogradov algebroids  $\mathcal{V}_n(M)$

- Locally as a **vector bundle**:  $\mathcal{V}_n(M) := T^*[n]T[1]M$
  - coords:  $(x^\mu, \xi^\mu, \zeta_\mu, p_\mu)$  of **degrees**  $(0, 1, n-1, n)$
  - Homological vector field:  $Q = \xi^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \zeta_\mu}$
  - Symplectic form:  $\omega = dx^\mu \wedge dp_\mu + d\xi^\mu \wedge d\zeta_\mu$
- 
- **Geometric picture**: principal  $n$ -bundle or  $n-1$ -gerbe
  - **diffeos+gauge**: degree  $n-1$  functions of  $\mathcal{V}_n(M)$ .
  - Higher **Courant/Dorfman brackets**
  - **Full Symmetrie Lie  $n$ -algebra**: Lie  $n$ -algebra of  $\mathcal{V}_n(M)$

How to extend this to Double Field Theory?



## Part II : Double Field Theory / Extended Geometry

- Generalized  $NQ$ -manifolds
- Construct example suitable for DFT
- Weakened section condition as algebraic relation
- Comments on global picture

There are two properties that offer themselves to a weakening.

## Requirements

- Geometry built from (doubled) **spacetime**
- **Symmetry Lie  $n$ -algebra structure** from derived brackets
- **Reduction** to Vinogradov algebroids of Generalized Geometry
- a few further points, related to **global picture**

Need to keep: **graded vector bundle**, **symplectic form** (?),  $|Q| = 1$

Note:  $Q^2 = 0$  **not necessary everywhere** for  $L_\infty$ -algebra:

$$L_{n-1}(\mathcal{M}) \xrightarrow{Q} L_{n-2}(\mathcal{M}) \xrightarrow{Q} \dots \xrightarrow{Q} L_1(\mathcal{M}) \xrightarrow{Q} L_0(\mathcal{M}) \xrightarrow{0} 0$$

$\searrow Q$   
 $C_n^\infty(\mathcal{M})$

Instead: something like  $\{Q^2 -, -\} = 0$ .

All relevant notions can be reasonably extended.

**Definition:** Symplectic pre-NQ-Manifold of degree  $n$

Symplectic N-manifold  $(\mathcal{M}, \omega)$  of degree  $n$ , i.e.  $|\omega| = n$ , compatible vector field  $Q$  of degree 1, i.e.  $|Q| = 1$  and  $\mathcal{L}_Q \omega = 0$ .

**Definition:**  $L_\infty$ -structure

A subset  $L(\mathcal{M})$  of the functions  $\mathcal{C}^\infty(\mathcal{M})$  such that the derived brackets close and form an  $L_\infty$ -algebra.

**Theorem**

$(\mathcal{M}, \omega)$  of degree 2.

$L(\mathcal{M}) = L_1(\mathcal{M}) \oplus L_0(\mathcal{M}) \subset \mathcal{C}^\infty(\mathcal{M})$ , derived brackets close.

$L(\mathcal{M})$  is  $L_\infty$ -structure iff for all  $f, g \in L_1(\mathcal{M})$ ,  $X, Y, Z \in L_0(\mathcal{M})$ :

$$\{Q^2 f, g\} + \{Q^2 g, f\} = 0, \quad \{Q^2 X, f\} + \{Q^2 f, X\} = 0$$

$$\{\{Q^2 X, Y\}, Z\}_{[X, Y, Z]} = 0$$

# Application to DFT: Built pre-NQ-manifold

The pre-NQ-manifold for DFT from a kind of polarization.

- Start:  $M = \mathbb{R}^D$ , double:  $T^*M$ , coords.  $x^M = (x^\mu, x_\mu)$
- $\mathcal{V}_2(T^*M) = T^*[2]T[1](T^*M)$ , coords.  $(x^M, \xi^M, \zeta_M, p_M)$
- $\omega = dx^M \wedge dp_M + d\xi^M \wedge d\zeta_M$ ,  $Q = \sqrt{2}(\xi^M \frac{\partial}{\partial x^M} + p_M \frac{\partial}{\partial \zeta_M})$
- Crucial to GenGeo/DFT: Local symmetry group  $\mathbf{O}(D, D)$
- Reduce structure group  $\mathbf{GL}(2D, \mathbb{R})$  to  $\mathbf{O}(D, D)$  by introducing

$$\eta_{MN} = \eta^{MN} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

- New coordinates:

$$\theta^M = \frac{1}{\sqrt{2}}(\xi^M + \eta^{MN}\zeta_N) \quad \beta^M = \frac{1}{\sqrt{2}}(\xi^M - \eta^{MN}\zeta_N)$$

- Polarize by putting  $\beta^M \stackrel{!}{=} 0$ :

$$Q = \theta^M \frac{\partial}{\partial x^M} + p_M \eta^{MN} \frac{\partial}{\partial \theta^N}, \quad Q^2 = p_M \eta^{MN} \frac{\partial}{\partial x^N} \neq 0$$

- We obtain a symplectic pre-NQ-manifold,  $\mathcal{E}_2(M)$ .

# Application to DFT: Weakened Section Condition

This framework is readily applied to reproduce DFT's strong section condition.

$$\mathcal{E}_2(\mathbb{R}^D) = \mathbb{R}^{2D} \oplus \mathbb{R}^{2D}[1] \oplus \mathbb{R}^{2D}[2]$$
$$x^M \qquad \theta^M \qquad p_M$$

$$Q = \theta^M \frac{\partial}{\partial x^M} + p_M \eta^{MN} \frac{\partial}{\partial \theta^N} \quad \omega = dx^M \wedge dp_M + \frac{1}{2} \eta_{MN} d\theta^M \wedge d\theta^N$$

## Proposition

Let  $L = L_0 \oplus L_1$  be  $L_\infty$ -structure on  $\mathcal{E}_2(\mathbb{R}^D)$ ,  $f, g \in L_1$  and  $X = X_M \theta^M, Y = Y_M \theta^M, Z = Z_M \theta^M \in L_0$ . Then:

$$\{Q^2 f, g\} + \{Q^2 g, f\} = 2\partial^M f \partial_M g = 0$$

$$\{Q^2 X, f\} + \{Q^2 f, X\} = 2\partial^M X \partial_M f = 0$$

$$\{\{Q^2 X, Y\}, Z\}_{[X,Y,Z]} = 2\theta^L ((\partial^M X_L)(\partial_M Y^K) Z_K)_{[X,Y,Z]} = 0$$

Note: Fulfilled for  $\partial^M \phi \partial_M \psi = 0 \Rightarrow$  Weakened section condition

Extended Geometry is sandwiched between Generalized Geometries.

The picture we have:

- Symmetries of DFT are described by  $L_\infty$ -structure on  $\mathcal{E}_2(\mathbb{R}^D)$
- Extended Geometry  $\mathcal{E}_2(\mathbb{R}^D)$ : polarization of GenGeo
- Reduce Ext. Geo.  $\rightarrow$  GenGeo: Choice of  $L_\infty$ -structure
- Example:

$$L = \left\{ F \in C^\infty(\mathcal{E}_2(\mathbb{R}^D)) \mid \frac{\partial}{\partial x_\mu} F = 0 \right\}$$
$$Q = \theta^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \theta^\mu} \quad \omega = dx^\mu \wedge dp_\mu + d\theta^\mu \wedge d\theta_\mu$$

$\Rightarrow$  Vinogradov algebroid  $\mathcal{V}_2(\mathbb{R}^D)$  of Generalized Geometry

An action of the symmetry Lie  $n$ -algebra can be defined.

Observations:

- Action of Lie algebra  $\mathfrak{g}$  on manifold  $M$ : hom.  $\mathfrak{g} \rightarrow \mathfrak{X}(M)$
- $N$ -manifold  $\mathcal{M}$ ,  $\mathfrak{X}(\mathcal{M})$  is  $\mathbb{N}_0$ -graded Lie algebra,  $L_\infty$ -algebra

Definition

Action of  $L_\infty$ -algebra  $L$  on manifold  $\mathcal{M}$ :  $L_\infty$ -morph.  $L \rightarrow \mathfrak{X}(\mathcal{M})$ .

Note:  $\mathcal{M} = \mathcal{V}_2(M)$  only encodes forms, not symmetric tensors.

Extension of Poisson bracket

$$\{-, -\} : C^\infty(\mathcal{M}) \times T(\mathcal{M}) \rightarrow T(\mathcal{M})$$
$$\{f, g \otimes h\} := \{f, g\} \otimes h + (-1)^{(n-|f|)|g|} g \otimes \{f, h\}$$

Definition: Extended tensors

Let  $L$  be  $L_\infty$ -structure on  $\mathcal{M}$ . Extended tensors are elts. of  $T(\mathcal{M})$  such that elements of  $L$  act on it via  $X \triangleright t := \{\delta X, t\}$ .

The global picture seems to be in reach now.

- **Surjective submersion**  $Y \twoheadrightarrow M$ ,  $Y^{[2]} := Y \times_M Y$ ,  $Y^{[n]} := \dots$
- For example,  $Y = \sqcup_i U_i$ ,  $Y^{[2]} = \sqcup_{i,j} U_i \cap U_j$
- Gerbe is a principal  $U(1)$ -bundle over  $Y^{[2]}$  + data over  $Y^{[3]}$
- Trivial gerbe:  $Y = M$ ,  $Y^{[2]} = M$ ,  $\mathcal{G} = M \times U(1) = M \times S^1$ .
- T-duality on  $M \times S^1$ : trivial gerbe,  $H = dB$  globally.

## Non-trivial gerbe in Generalized Geometry

- $\mathcal{V}_2(M) = T^*[2]T[1]M$  and  $Q = \xi^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \zeta_\mu}$  only **locally**
- Assume **nontrivial gerbe**  $H, B_{(i)}, A_{(ij)}, h_{(ijk)}$
- Hitchin's **Generalized tangent bundle** over patches  $i, j$ :  
 $(X_{(i)}, \Lambda_{(i)}) = (X_{(j)}, \Lambda_{(j)} + \iota_X dA_{(ij)})$
- $Q = \xi^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \zeta_\mu} - \frac{1}{3!} \left( \frac{\partial}{\partial x^\nu} H \right) \frac{\partial}{\partial p_\nu} + \frac{1}{2!} H_{\nu\mu_1\mu_2} \xi^{\mu_1} \xi^{\mu_2} \frac{\partial}{\partial \zeta_\nu}$



The global picture seems to be in reach now.

- $L_\infty$ -algebra of symmetries of DFT acts as Lie algebra
- Lie algebra can be integrated Hohm, Zwiebach, 2012
- Proposal: Patch local descriptions by finite DFT symmetries  
Berman, Cederwall, Perry, 2014
- Papadopoulos 2014: This only works for trivial gerbes
- No surprise, DFT reduces to GenGeo, where we need to twist!
- Need to twist C-/D-bracket, just as in GenGeo.
- Twist can be defined and studied in our framework.
- We recover twists of Generalized Geometry as special cases.
- Integrate twisted action
- Global picture: Patch together with twisted transformations!
- Understand all this  $\Rightarrow$  potentially win Berman medal

## Summary:

- ✓ Full **algebraic** and **geometric picture** for local DFT
- ✓ Picture is **extension** from GR coupled to  $n$ -form fields
- ✓ **weakened** section condition from algebra
- ✓ **twist** of symmetry Lie 2-algebra
- ✓ Initial studies of **global picture** and **Riemannian Geometry**

## More:

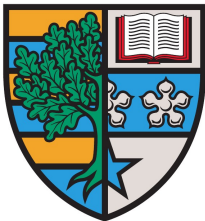
⇒ **Andreas' talk** this afternoon

## Soon to come:

- ▷ **Heterotic DFT**
- ▷ **Exceptional Field Theory** (M-theory)
- ▷ **Global Picture**

# Extended Riemannian Geometry and Double Field Theory

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