Pre-NQ-manifolds and derived brackets: Glimpses on torsion/curvature

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Preliminary words

After so many excellent talks, not much is needed to motivate the importance of **Courant algebroids**. Just some points:

- The current algebra of $2d$ WZW-models is governed by the Courant bracket Alekseev, Strobl
- Type II supergravity can be understood as Einstein-type gravity w.r.t an $O(9,1) \times O(1,9)$ structure on the generalized tangent bundle Coimbra, Graña, Minasian, Petrini, Strickland-Constable, Waldram
- Generalizations of Courant algebroids (Leibniz algebroids) found in M-theory Baraglia, Berman, Hull, Perry
- Notion of connections on Courant algebroids for heterotic string Jurco, Vysoky
- Poisson-Lie T-duality deals with Courant algebroids Klimcik, Ševera
- The C-brackets in double field theory/exceptional field theory are generalizations of the Courant bracket Berman, Perry, Hohm, Hull, Zwiebach
- (Membrane) Sigma models with Courant algebroids as target are used in deformation quantization as a route to non-associative gravity Aschieri, Blumenhagen, Fuchs, Mylonas, Schupp, Szabo

Preliminary words

- Lie 2-algebras for higher gauge theories Jurco, Ritter, Sämann, Schmidt, Wolf
- (Universal) sigma model for gauging along foliations has a Courant algebroid target Chatzistavrakidis, A.D., Jonke, Strobl
- Many more aspects and authors, I apologize for being very incomplete...

We use two results as motivation for our work:

Theorem 1 (Roytenberg, Weinstein).

Sections in a Courant algebroid give rise to a sh Lie algebra structure with vanishing n-ary maps for $n > 3$.

By using Poisson brackets on $T^*T[1]M$, Roytenberg showed

Theorem 2 (Roytenberg).

The Courant bracket is a derived bracket.

Recall: (Pre)-NQ-manifolds and derived brackets

Motivation: An easy calculation...

Recall from Christian's talk: Given a manifold M, consider $T[1]M$ with local coordinates (x^μ, ξ^μ). Its cotangent bundle $T^*T[1]M$ locally has $(x^\mu, \xi^\mu, p_\mu, \xi^*_\mu)$ and is Poisson:

$$\{p_\mu, x^\nu\} = \delta^\nu_\mu \qquad \{\xi^*_\mu, \xi^\nu\} = \delta^\nu_\mu \ .$$

Let's take the operator $Q = \xi^\mu p_\mu$, and vector fields $X = X^\mu \xi^*_\mu$, $Y = Y^\nu \xi^*_\nu$, then we can do the following exercise:

$$\begin{aligned}
\Big\{\{Q, X\}, Y\Big\} &= \Big\{\{\xi^\mu p_\mu, X^\nu \xi^*_\nu\}, Y^\rho \xi^*_\rho\Big\} \\
&= \Big\{\xi^\mu \partial_\mu X^\nu \xi^*_\nu + X^\nu p_\nu, Y^\rho \xi^*_\rho\Big\} \\
&= - Y^\rho \partial_\rho X^\nu \xi^*_\nu + X^\rho \partial_\rho Y^\nu \xi^*_\nu \\
&= [X, Y]^\nu_{\mathrm{Lie}} \xi^*_\nu \ .
\end{aligned}$$

We say, that the Lie bracket is a **derived bracket** (due to KOSMANN-SCHWARZBACH, ROYTENBERG, VORONOV).

Recall: (Pre)-NQ-manifolds and derived brackets

Important definitions

Definition 1.

A **symplectic pre-NQ-manifold of $\mathbb{N}$-degree** n *is an $\mathbb{N}$-graded manifold $\mathcal{M}$, together with symplectic form ω of degree n and a vector field Q of degree 1, satisfying $L_Q \omega = 0$.*

Examples

An important class where in addition $Q^2 = 0$, are the **Vinogradov Lie** n**-algebroids**:

$$\mathcal{V}_n(M) := T^*[n]T[1]M \ .$$

They have the following properties:

- Local coordinates $(x^\mu, \xi^\mu, \zeta_\mu, p_\mu)$ of degrees $0, 1, n-1, n$.
- Symplectic form $\omega = dx^\mu \wedge dp_\mu + d\xi^\mu \wedge d\zeta_\mu$
- Nilpotent vector field Q with Hamiltonian $\mathcal{Q} = \xi^\mu p_\mu$, i.e. $\{\mathcal{Q}, \mathcal{Q}\} = 0$.

Conditions for L_∞-structure

A.D., Sämann

If $Q^2 = 0$, the above brackets form an L_∞ structure. In our case we want to investigate conditions that this is also true, especially for $n = 2$, where we found the following

Theorem 3.

Consider the subset of $\mathcal{C}^\infty(\mathcal{M})$ consisting of functions and extended vector fiels, i.e. $\mathcal{C}_0^\infty(\mathcal{M}) \oplus \mathcal{C}_1^\infty(\mathcal{M})$. If the Poisson brackets and the maps μ_i close on this subset, the latter is an L_∞-algebra if and only if

$$\begin{aligned}
\{Q^2 f, g\} + \{Q^2 g, f\} &= 0\,, \\
\{Q^2 X, f\} + \{Q^2 f, X\} &= 0\,, \\
\{\{Q^2 X, Y\}, Z\}_{[X,Y,Z]} &= 0\,,
\end{aligned}$$

for all functions f, g and extended vector fields X, Y, Z. The notation $Q^2 f$ means $\{\mathcal{Q}, \{\mathcal{Q}, f\}\}$ and the subscript $[X, Y, Z]$ means the alternating sum over X, Y, Z.

The Courant bracket as a derived bracket

Roytenberg, Weinstein

For a manifold M, take $\mathcal{V}_2(M)$. Locally, coordinates are $(x^\mu, \xi^\mu, \zeta_\mu, p_\mu)$ of degrees $0, 1, 1, 2$. We get

- $\mathcal{Q} = \xi^\mu p_\mu$ squares to zero.
- Extended vectors, i.e. degree 1 objects, are now the "generalized vectors", i.e. $V = X^\mu \zeta_\mu + \alpha_\mu \xi^\mu$, $W = Y^\mu \zeta_\mu + \beta_\mu \xi^\mu$.
- For $f \in \mathcal{C}^\infty(M)$, $\{\mathcal{Q}, f\}$ gives the de Rham differential.
- For vectors V, W we get $\{\{\mathcal{Q}, V\}, W\} - V \leftrightarrow W = [X, Y]^\mu \zeta_\mu + (L_X\beta - L_Y\alpha - \frac{1}{2}d(\iota_X\beta - \iota_Y\alpha))_\mu \xi^\mu$, i.e. we get the Courant bracket.
- μ_3 (see Christian's talk) gives the the defect to the Jacobi identity for Courant algebroids.

So we recover generalized geometry on a Courant algebroid.

$n = 2$: Interpretation of the C-bracket

A.D., Sämann, Stasheff

We take the same setting as before, but instead of M as base, we take T^*M, i.e. we take $\mathcal{V}_2(T^*M)$. Local coordinates are now $(x^M, \xi^M, \zeta_M, p_M)$ of degree $(0, 1, 1, 2)$.

Problem: We now have too many "vectors". We solve this by defining

$$\theta^M := \tfrac{1}{\sqrt{2}}(\xi^M + \eta^{MN}\zeta_N) \quad \text{and} \quad \beta^M := \tfrac{1}{\sqrt{2}}(\xi^M - \eta^{MN}\zeta_N) \ ,$$

and taking only θ^M as degree-1 coordinates. Taking

$$\omega = dx^M \wedge dp_M + \tfrac{1}{2}\eta_{MN} d\theta^M \wedge d\theta^N, \quad \mathcal{Q} = \theta^M p_M \ ,$$

we get a pre-NQ-manifold (as $\mathcal{Q}$ doesn't square to zero, but we have $L_Q\omega = 0$ for the corresponding vector field).

$n = 2$: Interpretation of the C-bracket

A.D., Sämann, Stasheff

With this we get the following results:

- $\mu_1(f) = \theta^M \partial_M f$, i.e. the de Rham differential on the doubled space.
- For vectors $X = X_M \theta^M$, $Y = Y_M \theta^M$ we get, using $\eta^{MN} X_M \partial_N = X^N \partial_N$ etc.

  $$\mu_2(X, Y) = (X^M \partial_M Y_K - Y^M \partial_M X_K - \tfrac{1}{2}(Y^M \partial_K X_M - X^M \partial_K Y_M))\theta^K ,$$

  i.e. the C-bracket of double field theory.
- μ_3 gives the defect to the Jacobi identity of the C-bracket.

$n = 2$: Interpretation of the C-bracket

So locally the formulas give us double field theory, but what about the constraints for Q^2?

$n = 2$: Interpretation of the C-bracket

A.D., Sämann

To have a proper L_∞-structure, we still have to implement the constraints of our theorem. What are they? Let f be a function and $X = X_M\theta^M$, Y and Z be extended vectors.

- $\{Q^2 f, g\} + \{Q^2 g, f\} = 2\partial_M f\, \eta^{MN}\partial_N g = 0$
  This is the strong constraint.
- $\{Q^2 X, f\} + \{Q^2 f, X\} = 2(\partial_M X_K\theta^K)\eta^{MN}\partial_N f = 0$
  This is the strong constraint for vectors and functions.
- $\{\{Q^2 X, Y\}, Z\}_{[X,Y,Z]} = 2\theta^K((\partial^M X_K)(\partial_M Y^N)Z_N)_{[X,Y,Z]} = 0$
  Additional constraint for vectors? ...Shows up in properties of the Riemann tensor of double field theory

So the strong constraint together with the third constraint ensure the L_∞-structure for vectors and functions.

Definition 2.

An **extended covariant derivative** ∇ *on a pre-NQ-manifold* $\mathcal{M}$ *is a linear map from the set* $\mathcal{X}(\mathcal{M})$ *of extended vectors to* $\mathcal{C}^\infty(\mathcal{M})$*, such that the image* ∇_X *for* $X \in \mathcal{X}(\mathcal{M})$ *gives a map* $\{\nabla_X, \cdot\}: \mathcal{X}(\mathcal{M}) \to \mathcal{X}(\mathcal{M})$*, which satisfies*

$$\{\nabla_{fX}, Y\} = f\{\nabla_X, Y\} \quad \mathit{and} \quad \{\nabla_X, fY\} = \{\{\mathcal{Q}, X\}, f\}Y + f\{\nabla_X, Y\}\,,$$

for all functions f *and extended vectors* Y*. For arbitrary extended tensors extend this by the graded Leibniz rule of the Poisson bracket*

$$\{V, W \otimes U\} := \{V, W\} \otimes U + (-1)^{(n-|W|)|U|} W \otimes \{V, U\}\,,$$

where $V, W, U \in \mathcal{C}^\infty(\mathcal{M})$ *and* $|W|$ *denotes the degree.*

Some remarks

- For any $\mathcal{V}_n$, we have a Cartan calculs with generators

$$L_X := \{QX, \cdot\}\,, \quad \iota_X := \{X, \cdot\}\,, \quad d := \{Q, \cdot\}\,.$$

Exterior covariant calculus? Work in progress...

- Pointwise, $\mathcal{X}$ is a vector space, so we can consider also its dual. For torsion and curvature we use $\hat{\mathcal{X}} := \mathcal{X} \oplus \mathcal{X}^*$.
- We also denote by $\pi : \hat{\mathcal{X}} \to \mathcal{X}$ the projection to the first summand.
- In the following we will deal with $\mathcal{V}_1$ and the restricted $\mathcal{V}_2$ suitable for double field theory. In these cases, one can show that the following functions have the right properties:

$$\begin{aligned}\nabla_X &= X^\mu p_\mu - X^\mu \Gamma^\rho{}_{\mu\nu} \zeta_\rho \xi^\nu\ ,\\ \nabla_X &= X^M p_M - \tfrac{1}{2} X^M \Gamma_{MNK} \theta^N \theta^K\ .\end{aligned}$$

Definition 3.
Let $\mathcal{M}$ be a pre-NQ-manifold. Given an extended connection ∇, we define the **extended torsion tensor** $\mathcal{T} : \otimes^3 \hat{\mathcal{X}}(\mathcal{M}) \to \mathcal{C}^\infty(M)$ *for* $X, Y, Z \in \hat{\mathcal{X}}(\mathcal{M})$ *by*

$$\mathcal{T}(X,Y,Z) := 3\,Alt_{XYZ}\Big((-1)^{n|X|}\left\{X, \{\nabla_{\pi(Y)}, Z\}\right\} + \frac{(-1)^{n(|Y|+1)}}{2}\left(\{X,\{QZ,Y\}\} - \{Z,\{QX,Y\}\}\right)\Big) \ ,$$

where $|X|, |Y|$ denote the respective degrees, π is the above defined projection and $n = 1, 2$ is the degree of the underlying Vinogradov algebroid.

With this definition, we are able to show the following results relating extended torsion to standard ones:

Theorem 4.
The extended torsion is $\mathcal{C}^\infty(M)$-linear in every entry ($n = 1, 2$). For $\mathcal{M} = \mathcal{V}_1(M)$, let $X \in \mathcal{X}^*(\mathcal{M})$ and $Y, Z \in \mathcal{X}(\mathcal{M})$, then the extended torsion reduces to the torsion operator $T(X, Y, Z) = \langle X, \nabla_Y Z - \nabla_Z Y - [Y, Z]\rangle$, where the bracket is the Lie bracket of vector fields. More generally, this is true whenever we take one element of $\mathcal{X}^*(\mathcal{M})$ and the other two in $\mathcal{X}(\mathcal{M})$. In all other cases the extended torsion vanishes. In case of double field theory and $\mathcal{V}_2$, for extended vector fields X, Y, Z, the extended torsion tensor equals the Gualtieri torsion of generalized geometry.

Definition 4.

Let $\mathcal{M}$ be a pre-NQ-manifold. Given an extended connection ∇, the **extended curvature operator** $\mathcal{R} : \otimes^4 \hat{\mathcal{X}}(\mathcal{M}) \to \mathcal{C}^\infty(M)$ *for* $X, Y, Z, W \in \hat{\mathcal{X}}(\mathcal{M})$ *is defined by*

$$
\begin{aligned}
\mathcal{R}(X,Y,Z,W) :=& \\
&\frac{1}{2}\Big(\Big\{\{\{\nabla_X, \nabla_Y\} - \nabla_{\mu_2(X,Y)}, Z\}, W\Big\} - (-1)^n (Z \leftrightarrow W) \\
&+ \Big\{\{\{\nabla_Z, \nabla_W\} - \nabla_{\{\nabla_Z, W\} - \{\nabla_W, Z\}}, X\}, Y\Big\} - (-1)^n (X \leftrightarrow Y)\Big) .
\end{aligned}
$$

Reminder: μ_2 is the C-bracket in the derived-bracket form.

Theorem 5.
For $\mathcal{M} = \mathcal{V}_1(M)$, let $X, Y, Z \in \mathcal{X}(\mathcal{M})$ and $W \in \mathcal{X}^*(\mathcal{M})$. Then the extended curvature reduces to the standard curvature:

$$\mathcal{R}(X, Y, Z, W) = \langle W, \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \rangle .$$

Furthermore, if $X, Y, Z, W \in \mathcal{X}(\mathcal{M})$ or if two, three or all of X, Y, Z, W are in $\mathcal{X}^*(\mathcal{M})$, we have $\mathcal{R}(X, Y, Z, W) = 0$. Moreover, for vanishing extended torsion, in case of double field theory the extended curvature is the Hohm-Zwiebach curvature. In this case, $\mathcal{C}^\infty(M)$-linearity holds by the constraints given in theorem 3.

It is the last sentence, where the algebraic setting becomes important for geometry.

Outlook

What we did...

We found a unifying language to describe the Lie bracket, Courant bracket and C-bracket. The strong constraint plays a role to ensure an L_∞-structure on functions and vectors.

- In all three cases, arbitrary tensors can be defined, extended Lie derivatives and the action of infinitesimal extended diffeomorphisms (see Christian's talk!)
- We are lead towards a "derived geometry", including torsion, Gualtieri-torsion and Riemann tensors (so far for $n = 1, 2$). This is very preliminary and still ongoing work! Math question: *Is there a general notion of torsion and curvature for any Vinogradov algebroid?*

Outlook

Open questions

- Everything was local. Global analysis? Gerbes, groupoids... What is the global description of double field theory? T-duality?
- For higher Vinogradov algebroids $\mathcal{V}_n(M)$, degree $n-1$-objects are

  $$X = X^\mu \zeta_\mu + X_{\mu_1 \ldots \mu_{n-1}} \xi^{\mu_1} \cdots \xi^{\mu_{n-1}} ,$$

  i.e. sections of $TM \oplus \wedge^{n-1} T^* M$. For $n = 3$, we get the easiest case of *exceptional generalized geometry*, where the U-duality group is $SL(5, \mathbb{R})$. How about the other exceptional tangent bundles?
- What are torsion and Riemann tensors in exceptional generalized geometries? Do they have a meaning in Poisson geometry on certain Vinogradov algebroids?
- Quantization: If we can write the brackets in terms of Poisson brackets, we can do deformation quantization!

Appendix: Remark about quantization

A.D, in discussion with J.Stasheff

In string theory/double field theory, higher derivative corrections (i.e. corrections in the sigma model coupling α') to the bilinear form η and the C-bracket were computed by Hohm, Zwiebach:

$$\langle V, W\rangle_{\alpha'} = \langle V, W\rangle - \alpha' \, \partial_P V^Q \partial_Q W^P \ ,$$

$$[V, W]^K_{\alpha'} = [V, W]^K_C - \alpha' \Big(\partial^K \partial_Q V^P \partial_P W^Q - V \leftrightarrow W\Big) \ .$$

Idea: We constructed the C-bracket in terms of Poisson brackets. Instead of Poisson brackets, take the star-commutator. For (M, π) with constant π, take the standard Moyal-Weyl product for the constant Poisson structure:

$$P_{T^*T[1]M} = \partial_{p_\mu} \wedge \partial_{x^\mu} + \partial_{\zeta_\mu} \wedge \partial_{\xi^\mu} + \partial_{x^\mu} \wedge \partial_{\zeta_\mu} + \pi^{\mu\nu} \partial_{x^\nu} \wedge \partial_{\xi^\mu} \ .$$

Appendix: Remark about quantization

A.D, in discussion with J.Stasheff

Theorem 6.
Let $V = V^\mu \zeta_\mu + V_\mu \xi^\mu$ *and* $W = W^\mu \zeta_\mu + W_\mu \xi^\mu$ *be the lifts of two generalized vectors to* $T^*T[1]M$*. Then we have*

$$\frac{1}{\alpha'}\{V, W\}^\star = \langle V, W\rangle_{\alpha'} + \mathcal{O}((\alpha')^2) \; .$$

Furthermore, we have

$$\frac{1}{2(\alpha')^2}\Big(\big\{\{\mathcal{Q}, V\}^*, W\big\}^* - \big\{\{\mathcal{Q}, W\}^*, V\big\}^*\Big) = [V, W]_{\alpha'} + \mathcal{O}((\alpha')^2) \; ,$$

i.e. the α'*-deformations encountered in string theory can be understood in terms of appropriate star commutators.*