# Generalized/higher geometry and non-associative quantum mechanics

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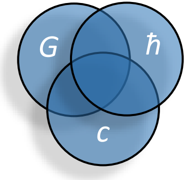
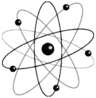
cost Action MP 1405
Quantum Structure of Spacetime

## Outline

- Introduction
- Fluxes and generalized/higher geometry
- Interaction via deformation
- Nonassociativity and quantum mechanics

# Introduction: Quantized Geometry



## Quantum spacetime

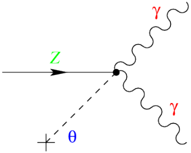
*quantum + gravity* $\Rightarrow$

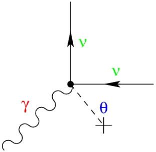
Quantized geometry: *apply the principles of QM to spacetime itself*

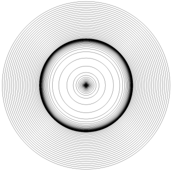
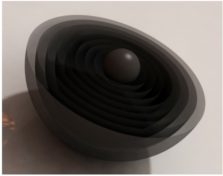
- microscopic non-commutative/non-associative spacetime structures
- expect spacetime coarse-graining, natural regularization

## Quantum fields on noncommutative spaces

- $[x^i, x^j] = i\theta^{ij} \neq 0$
- forbidden interactions, controlled Lorentz violation, UV/IR mixing
- NC Standard Model, NC GUTs, etc.
- Gravity on noncommutative spaces [, ] ≠ 0
  twisted tensor calculus, deformed Einstein equations





All very interesting. . . but:

- no space-time coarse graining!
- $\theta \sim B^{-1}$, how to deal with $dB = H \neq 0$?

$\Rightarrow$ higher structures, deformations

# Strings and generalized geometry: non-geometric fluxes

## Non-geometric flux backgrounds

T-dualizing a 6-torus with 3-form $H$-flux gives rise to geometric and

non-geometric fluxes $\quad H_{ijk} \xrightarrow{T_k} f_{ij}{}^{k} \xrightarrow{T_j} Q_i{}^{jk} \xrightarrow{T_i} R^{ijk}$

Hellermann, McGreevy, Williams (2004)
Hull (2005), Shelton, Taylor, Wecht (2005)
Lüst (2010), Blumenhagen, Plauschinn (2010)

Generalized (doubled) geometry ($O(d,d)$ isometry, $g$, $B$,...)

Non-geometry geometrized in membrane model
quantization $\Rightarrow$ non-associative $\star$-product

Mylonas, PS, Szabo (2012-2013)

# Strings and generalized geometry: non-geometric fluxes

| | |
|---|---|
| $H_{ijk}$ | 3-form background flux |
| ${f_{ij}}^k$ | geometric flux, $[e_i, e_j]_L = {f_{ij}}^k e_k$ |
| $Q_i^{jk}$ | globally non-geometric, T-fold |
| $R^{ijk}$ | locally non-geometric, non-associative |

structure constant of a generalized bracket:

$$
\begin{aligned}
[e_i, e_j]_C &= {f_{ij}}^k e_k + H_{ijk} e^k \\
[e_i, e^j]_C &= Q_i^{jk} e_k - {f_i}^j{}_k e^k \\
[e^i, e^j]_C &= R^{ijk} e_k + Q^{ij}{}_k e^k
\end{aligned}
$$

Courant/Dorfman/Roytenberg bracket on $\Gamma(TM \oplus T^*M)$
governs worldsheet current and charge algebras

Alekseev, Strobl; Halmagyi; Bouwknegt; . . .

## Dorfman bracket

Generalizes the Lie bracket of vector fields $X \in \Gamma(TM)$ to $V = X + \xi \in \Gamma(TM \oplus T^*M)$:

$$[X + \xi, Y + \eta]_D = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi \quad \text{(+twisting terms)}$$

$E = TM \oplus T^*M$ is called "generalized tangent bundle"

$E$ with the Dorfman bracket, the natural pairing $\langle -, - \rangle$ of $TM$ and $T^*M$ and the projection $h : E \to TM$ (anchor) forms a Courant algebroid.

## Courant algebroid

vector bundle $E \xrightarrow{\pi} M$, anchor $h \in \mathrm{Hom}(E, TM)$,
$\mathbb{R}$-bilinear bracket $[-,-]$ , and fiber-wise metric $\langle -,- \rangle$,
s.t. for $e, e', e'' \in E$:

$$[e,[e',e'']] = [[e,e'],e''] + [e',[e,e'']] \tag{1}$$

$$h(e)\langle e',e'\rangle = 2\langle [e,e'],e'\rangle = 2\langle [e',e'],e\rangle \tag{2}$$

Consequences:

$$[e, fe'] = h(e).f\, e' + f[e,e'] \tag{3}$$

$$h([e,e']) = [h(e),h(e')]_L \tag{4}$$

(2) can be polarized
(1) and (3) are the axioms of a Leibniz algebroid

## Graded "super" Poisson manifold $T^*[2]T[1]M$

- degree 0: $x^i$ "coordinates"
- degree 1: $\xi^\alpha = (\theta^i, \chi_i)$
- degree 2: $p_i$ "momenta"

symplectic 2-form

$$\omega = dp_i \wedge dx^i + \frac{1}{2} G_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta = dp_i \wedge dx^i + d\chi_i \wedge \theta^i + d\theta_i \wedge \chi^i$$

even (degree -2) Poisson bracket

$$\{x^i, x^j\} = 0, \quad \{p_i, x^j\} = \delta_i^j, \quad \{\xi^\alpha, \xi^\beta\} = G^{\alpha\beta}$$

metric $G^{\alpha\beta}$: natural pairing of $TM$, $T^*M$

$$\{\chi_i, \theta^j\} = \delta_i^j \ , \quad \{\chi_i, \chi_j\} = 0 \ , \quad \{\theta^i, \theta^j\} = 0 \ ,$$

## Generalized geometry as a derived structure

Hamiltonian

$$\Theta = \xi^\alpha h^i_\alpha(x) p_i \quad (+\text{twisting terms})$$

For $e = e_\alpha(x)\xi^\alpha$ (degree 1, odd):

- pairing: $\langle e, e' \rangle = \{e, e'\}$
- anchor: $h(e)f = \{\{e, \Theta\}, f\}$
- bracket: $[e, e']_D = \{\{e, \Theta\}, e'\}$

$$\{\Theta, \Theta\} = 0 \qquad \Leftrightarrow \qquad \text{Courant algebroid axioms}$$

## Deformation and interaction I: gravity

deformation by a non-symmetric metric $\mathcal{G} = g + B$

$$\{\chi_i, \chi_j\} = 0 \quad \to \quad \{\chi_i, \chi_j\}' = 2g_{ij}(x)$$

$\Rightarrow$ for $X = X^i(x)\chi_i$ . $v = v^i(x)p_i$:

$$\{v, X\}' = \nabla^G_v X \ , \quad \{v, v'\}' = [v, v']_{\mathrm{Lie}} + R(v, v')$$

choose Weitzenböck connection $\Rightarrow R(v, v') = 0$ and

$$\nabla^G_i \chi_j = -(\partial_i \mathcal{G}_{jl})\, \theta^l$$

the derived bracket involves the Levi-Civita connection $\nabla^{\mathrm{LC}}$

$$[X, Y]' = [X, Y]_D + 2g(\nabla^{\mathrm{LC}} X, Y) + H(-, X, Y)$$

plus skew symmetric torsion $H = dB$.

# Generalized/higher geometry and gravity

generalized Koszul formula for $\mathcal{G} = g + B$

$$
\begin{aligned}
2g(\nabla_Z X, Y) &= \langle Z, [X, Y]' \rangle' \\
&= X\mathcal{G}(Y, Z) - Y\mathcal{G}(X, Z) + Z\mathcal{G}(X, Y) \\
&\quad -\mathcal{G}(Y, [X, Z]_{\mathrm{Lie}}) - \mathcal{G}([X, Y]_{\mathrm{Lie}}, Z) + \mathcal{G}(X, [Y, Z]_{\mathrm{Lie}}) \\
&= 2g(\nabla^{LC}_X Y, Z) + H(X, Y, Z)
\end{aligned}
$$

$\Rightarrow$ non-symmetric Ricci tensor

$$
R_{jl} = R^{LC}_{jl} - \frac{1}{2}\nabla^{LC}_i H_{jl}{}^i - \frac{1}{4} H_{lm}{}^i H_{ij}{}^m
$$

$\Rightarrow$ gravity action (= closed string effective action)

$$
S_{\mathcal{G}} = \frac{1}{16\pi G_N} \int d^d x \sqrt{-g} \left( R^{LC} - \frac{1}{12} H_{ijk} H^{ijk} \right)
$$

Jurco, Khoo, PS, Vysoky

## Deformation and interaction II: gauge theory

Note: $\vec{B} = \nabla \times \vec{A}$ implies $\nabla \cdot B = 0$, hence we cannot work with canonical momenta and covariant derivatives in the presence of magnetic sources.
alternatively: deformed canonical commutation relations

$$[x^i, x^j]' = 0 \ , \ [x^i, p_j]' = i\hbar \ , \ [p_i, p_j]' = i\hbar e F_{ij} \quad (\text{where } F_{ij} = \epsilon_{ijk} B_k)$$

Let $\mathbf{p} = p_i \sigma^i$ and $H = \dfrac{\mathbf{p}^2}{2m} \quad \Rightarrow \quad$ Pauli Hamiltonian:

$$H = \frac{1}{2m}\left(\frac{1}{4}\{\sigma^i, \sigma^j\}\{p_i, p_j\}' + \frac{1}{4}[\sigma^i, \sigma^j][p_i, p_j]'\right) = \frac{\vec{p}^{\,2}}{2m} - \frac{\hbar e}{2m}\vec{\sigma} \cdot \vec{B}$$

Lorentz-Heisenberg equations of motion:

$$\frac{d\vec{p}}{dt} = \frac{i}{\hbar}[H, \vec{p}]' = \frac{e}{2m}\left(\vec{p} \times \vec{B} - \vec{B} \times \vec{p}\right) \ , \quad \frac{d\vec{r}}{dt} = \frac{i}{\hbar}[H, \vec{r}]' = \frac{\vec{p}}{m}$$

in this formalism $\nabla \cdot B \neq 0$ is allowed (magnetic sources)

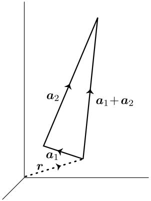
# QM with 3-cocycle

Jacobi identity:

$$[p_1,[p_2,p_3]']' + [p_2,[p_3,p_1]']' + [p_3,[p_1,p_2]']' = \hbar^2 e \nabla \cdot \vec{B} = \hbar^2 e \mu_o \rho_m$$

For $\rho_m \neq 0$: non-associativity, $\nexists$ linear operator $\vec{p} = -i\hbar\nabla - e\vec{A}$

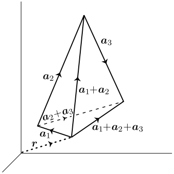
Translations are generated by $T(\vec{a}) = \exp(\frac{i}{\hbar}\vec{a}\cdot\vec{p})$:




$$T(\vec{a}_1)T(\vec{a}_2) = e^{\frac{ie}{\hbar}\Phi_{12}}T(\vec{a}_1+\vec{a}_2)$$

$$[T(\vec{a}_1)T(\vec{a}_2)]T(\vec{a}_3) = e^{\frac{ie}{\hbar}\Phi_{123}}T(\vec{a}_1)[T(\vec{a}_2)T(\vec{a}_3)]$$

$\Phi_{12}$ = flux through triangle $(\vec{a}_1,\vec{a}_2)$
$\Phi_{123}$ = flux out of tetrahedron $(\vec{a}_1,\vec{a}_2,\vec{a}_3) = \mu_0 q_m$




Associativity of translations is restored for:

$$\boxed{\frac{\mu_0 e q_m}{\hbar} \in 2\pi\mathbb{Z}}$$ (Dirac charge-quantization)

point-like magnetic monopoles . . . else: need NAQM

Jackiw '85,'02

# Aspects of quantization $\theta(x) \rightsquigarrow \star$

The operator-state formulation of QM cannot handle non-associative structures. . .

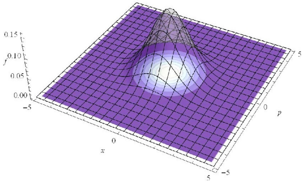
## Phase-space formulation of QM

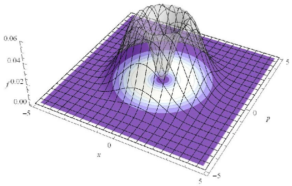
- Observables and states are (real) functions on phase space.
- Algebraic structure introduced by a star product,
  traces by integration.
- State function (think: "density matrix"): $S_\rho \geq 0$, $\int S_\rho = 1$.
- Expectation values $\langle \mathcal{O} \rangle = \int \mathcal{O} \star S_\rho$.
- Schrödinger equation $H \star S_\rho - S_\rho \star H = i\hbar \dfrac{\partial S_\rho}{\partial t}$
- "Stargenvalue" equation: $H \star S_\rho = S_\rho \star H = E\, S_\rho$.

# Aspects of quantization $\theta(x) \rightsquigarrow \star$

## Popular choices of star products

- *Moyal-Weyl* (symmetric ordering, Wigner quasi-probability function)
  Weyl quantization associates operators to polynomial functions via symmetric ordering: $x^\mu \rightsquigarrow \hat{x}^\mu$, $x^\mu x^\nu \rightsquigarrow \frac{1}{2}(\hat{x}^\mu \hat{x}^\nu + x^\nu \hat{x}^\mu)$, etc.
  extend to functions, define star product $\widehat{f_1 \star f_2} := \widehat{f_1}\,\widehat{f_2}$ .
- *Wick-Voros* (normal ordering, coherent state quantization)
  QHO states in Wick-Voros formulation:




- *xp-ordered star product:* $\star$-exponential $\equiv$ ordinary path integral

# Aspects of quantization $\theta(x) \rightsquigarrow \star$

Deformation quantization of the point-wise product in the direction of a Poisson bracket $\{f, g\} = \theta^{ij}\partial_i f \cdot \partial_j g$:

$$f \star g = fg + \frac{i\hbar}{2}\{f, g\} + \hbar^2 B_2(f, g) + \hbar^3 B_3(f, g) + \dots ,$$

with suitable bi-differential operators $B_n$.

There is a natural gauge symmetry: "equivalent star products"

$$\star \mapsto \star' , \quad Df \star Dg = D(f \star' g) ,$$

with $Df = f + \hbar D_1 f + \hbar^2 D_2 f + \dots$

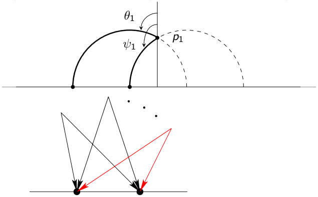
# Aspects of quantization $\quad \theta(x) \rightsquigarrow \star$

## Kontsevich formality and star product

$U_n$ maps $n$ $k_i$-multivector fields to a $(2-2n+\sum k_i)$-differential operator

$$U_n(\mathcal{X}_1,\ldots,\mathcal{X}_n) = \sum_{\Gamma\in G_n} w_\Gamma\, D_\Gamma(\mathcal{X}_1,\ldots,\mathcal{X}_n)\ .$$

The star product for a given bivector $\theta$ is:




$$\begin{aligned} f\star g =& \sum_{n=0}^{\infty} \frac{(\mathrm{i}\,\hbar)^n}{n!}\, U_n(\Theta,\ldots,\Theta)(f,g) \\ =& f\cdot g + \frac{i}{2}\sum \theta^{ij}\,\partial_i f\cdot\partial_j g - \frac{\hbar^2}{4}\sum \theta^{ij}\theta^{kl}\,\partial_i\partial_k f\cdot\partial_j\partial_l g \\ &- \frac{\hbar^2}{6}\Big(\sum \theta^{ij}\partial_j\theta^{kl}\ (\partial_i\partial_k f\cdot\partial_l g - \partial_k f\cdot\partial_i\partial_l g)\Big) + \ldots \end{aligned}$$

Kontsevich (1997)

## Formality condition

The $U_n$ define a quasi-isomorphisms of $L_\infty$-DGL algebras and satisfy

$$\mathrm{d}.U_n(\mathcal{X}_1,\ldots,\mathcal{X}_n)+\frac{1}{2}\sum_{\substack{\mathcal{I}\sqcup\mathcal{J}=(1,\ldots,n)\\ \mathcal{I},\mathcal{J}\neq\emptyset}}\varepsilon_{\mathcal{X}}(\mathcal{I},\mathcal{J})\left[U_{|\mathcal{I}|}(\mathcal{X}_{\mathcal{I}}),\,U_{|\mathcal{J}|}(\mathcal{X}_{\mathcal{J}})\right]_{\mathrm{G}}$$
$$=\sum_{i<j}(-1)^{\alpha_{ij}}\,U_{n-1}\big([\mathcal{X}_i,\mathcal{X}_j]_{\mathrm{S}},\mathcal{X}_1,\ldots,\widehat{\mathcal{X}}_i,\ldots,\widehat{\mathcal{X}}_j,\ldots,\mathcal{X}_n\big)\ ,$$

relating Schouten brackets to Gerstenhaber brackets.

This implies in particular $\Phi(\mathrm{d}_\Theta\Theta)=\frac{1}{\mathrm{i}\,\hbar}\mathrm{d}_\star\Phi(\Theta)$, i.e.

$\theta$ (non-)Poisson $\quad\Leftrightarrow\quad$ $\star$ (non-)associative

## Poisson sigma model

2-dimensional topological field theory, $E = T^*M$

$$S^{(1)}_{\mathrm{AKSZ}} = \int_{\Sigma_2} \Big(\xi_i \wedge \mathrm{d}X^i + \frac{1}{2}\,\Theta^{ij}(X)\,\xi_i \wedge \xi_j\Big) \ ,$$

with $\Theta = \frac{1}{2}\,\Theta^{ij}(x)\,\partial_i \wedge \partial_j$ , $\xi = (\xi_i) \in \Omega^1(\Sigma_2, X^*T^*M)$

perturbative expansion $\Rightarrow$ Kontsevich formality maps

valid on-shell ($[\Theta,\Theta]_S = 0$) as well as off-shell, e.g. twisted Poisson

Kontsevich (1997)
Cattaneo, Felder (2000)

# Higher geometry

geometric ladder / extended objects in background fields

| | | | |
|---|---|---|---|
| AKSZ-model: | Poisson-sigma<br>(open string)<br>$T^*[1]M$ | Courant-sigma<br>(open membrane)<br>$T^*[2]T[1]M$ | ... |
| derived bracket: | Poisson | Dorfman | ... |
| | • |  | |
| object: | point particle | closed string | ... |
| algebraic structure: | non-commutative | non-associative | ... |

# Geometrized non-geometry: membrane sigma model

## Courant sigma model

TFT with 3-dimensional membrane world volume $\Sigma_3$

$$S^{(2)}_{\mathrm{AKSZ}} = \int_{\Sigma_3} \Big( \phi_i \wedge \mathrm{d}X^i + \frac{1}{2}\, G_{IJ}\, \alpha^I \wedge \mathrm{d}\alpha^J - h_I{}^i(X)\, \phi_i \wedge \alpha^I + \frac{1}{6}\, T_{IJK}(X)\, \alpha^I \wedge \alpha^J \wedge \alpha^K \Big)$$

embedding maps $X : \Sigma_3 \to M$, 1-form $\alpha$, aux. 2-form $\phi$, fiber metric $G$, anchor $h$, 3-form $T$ (e.g. $H$-flux, $f$-flux, $Q$-flux, $R$-flux).

AKSZ construction: action functionals in BV formalism of sigma model QFT's for symplectic Lie $n$-algebroids $E$

Alexandrov, Kontsevich, Schwarz, Zaboronsky (1995/97)

# Geometrized non-geometry: membrane sigma model

## Membrane action with $R$-flux

$$S_R^{(2)} = \int_{\Sigma_3} \left( d\xi_i \wedge \mathrm{d}X^i + \frac{1}{6}\, R^{ijk}(X)\, \xi_i \wedge \xi_j \wedge \xi_k \right)$$

for constant backgrounds, using Stokes leads to boundary action

$$S_R^{(2)} = \int_{\Sigma_2} \left( \eta_I \wedge \mathrm{d}X^I + \frac{1}{2}\, \Theta^{IJ}(X)\, \eta_I \wedge \eta_J \right) \; :$$

Poisson sigma-model with auxiliary fields $\eta_I$ and

$$\Theta = (\Theta^{IJ}) = \begin{pmatrix} R^{ijk}\, p_k & \delta^i{}_j \\ -\delta_i{}^j & 0 \end{pmatrix} \quad \longrightarrow \quad \star \quad \text{(non-associative!)}$$

$$f \star g = \cdot \exp\left( \frac{i\hbar}{2} \left[ R^{ijk} p_k \partial_i \otimes \partial_j + \partial_i \otimes \tilde{\partial}^i - \tilde{\partial}^i \otimes \partial_i \right] \right)$$

Mylonas, PS, Szabo (2012)

# Jordan Algebra

## Noncommutative Jordan Algebras

$$(1) \qquad x(yx) = (xy)x \qquad \text{“flexible”}$$

$$(2) \qquad x^2(yx) = (x^2y)x$$

properties (1) and (2) imply

$$(3) \qquad x^m(yx^n) = (x^my)x^n \qquad \text{“power associative”}$$

and are necessary and sufficient conditions for

$$x \circ y := \frac{1}{2}(xy + yx)$$

to be Jordan, i.e. $x \circ y = y \circ x$ and $(x \circ y) \circ x^{\circ 2} = x \circ (y \circ x^{\circ 2})$.

P. Jordan (1933), A.A. Albert (1946), R.D. Schafer (1955)

# Jordan Algebra

non-associative star product

$$f \star g = \cdot \exp\left(\frac{i\hbar}{2}\left[R^{ijk} p_k \partial_i \otimes \partial_j + \partial_i \otimes \tilde{\partial}^i - \tilde{\partial}^i \otimes \partial_i\right]\right)$$

Question: Are we dealing with a Jordan algebra?

$$x^I \star (x^K \star x^I) = (x^I \star x^K) \star x^I \qquad \checkmark$$

$$(x^I)^{\star 2} \star (x^K \star x^I) = ((x^I)^{\star 2} \star x^K) \star x^I \qquad \checkmark$$

but:

$$\vec{x}^2 \star (\vec{x}^2 \star \vec{x}^2) - (\vec{x}^2 \star \vec{x}^2) \star \vec{x}^2 = 2iR^2 \vec{p} \cdot \vec{x} \neq 0$$

for $R^{ijk} \equiv R\epsilon^{ijk}$. $\Rightarrow$ Answer: no

Alexander Held, PS (2014), Bojowald, Brahma, Büyükcam, Strobl (2016)

# Günaydin-Zumino Model

Exchange $x$ and $p$, replace $R^{ijk}$ by $H_{ijk}$ ...

$$[x^i, p_j]_\star = i\delta^i_j \qquad [x^i, x^j]_\star = 0 \qquad [p_i, p_j]_\star = iH_{ijk}x^k$$

algebra of coordinates and physical (gauge invariant) momenta in a constant homogeneous magnetic charge density background

- coarse graining in momentum space
- three copies of $\vec{p}^2$ do not associate:

$$\vec{p}^2 \star (\vec{p}^2 \star \vec{p}^2) - (\vec{p}^2 \star \vec{p}^2) \star \vec{p}^2 = 2ie\rho^2_{\text{magnetic}}\vec{x}\cdot\vec{p} \neq 0$$

$\Rightarrow$ cannot diagonalize? $\Rightarrow$ no free stationary states??

- eigenfunctions: just need to make sure that $\langle \vec{x}\cdot\vec{p}\rangle = 0$, in fact:

$$p_i^2 \star \psi = \lambda_i \psi \quad \Rightarrow \quad \psi(x,p) \propto \exp(2ix^i(p_i - \lambda_i)), \qquad \lambda_i \in \mathbb{R}$$

# Nonassociative quantum mechanics

## Phase-space formulation of NAQM

- *Operators:* complex-valued functions on phase space – the star product serves as operator product
- *Observables:* real-valued functions on phase-space
- *Dynamics:* Heisenberg-type time evolution equations

$$\frac{\partial A}{\partial t} = \frac{i}{\hbar}[H, A]_\star$$

these are in general not derivations of the star product!

non-associative star product

$$f \star g = \cdot \exp\left(\frac{i\hbar}{2}\left[R^{ijk} p_k \partial_i \otimes \partial_j + \partial_i \otimes \tilde{\partial}^i - \tilde{\partial}^i \otimes \partial_i\right]\right)$$

Mylonas, PS, Szabo (2012-2013)

## Trace properties

- 2-cyclicity (trace-less commutator) , positivity

$$\int \mathrm{d}^{2d}x \; [f \star g - g \star f] = 0 \quad , \qquad \int \mathrm{d}^{2d}x \; f^* \star f \geq 0$$

- 3-cyclicity (trace-less associator)

$$\int \mathrm{d}^{2d}x \; [(f \star g) \star h - f \star (g \star h)] = 0$$

inequivalent quartic expressions

$$\int f_1 \star \big(f_2 \star (f_3 \star f_4)\big) = \int (f_1 \star f_2) \star (f_3 \star f_4) = \int \big((f_1 \star f_2) \star f_3\big) \star f_4$$

$$\int f_1 \star \big((f_2 \star f_3) \star f_4\big) = \int \big(f_1 \star (f_2 \star f_3)\big) \star f_4$$

# Nonassociative quantum mechanics

## Two conjugate associative algebras

- left and right compositions

$$(A \circ B) \star C := A \star (B \star C) \;, \qquad C \star (A \,\bar{\circ}\, B) := (C \star A) \star B$$

$$(A_1 \circ A_2 \circ \ldots \circ A_n) \star C = A_1 \star (A_2 \star \ldots (A_n \star C) \ldots))$$

- $A \circ 1 = A = 1 \circ A$
- $A \circ B$ is typically not a function; some notable exceptions:

$$x^i \circ x^i = x^i \star x^i = (x^i)^2 \qquad p_i \circ p_i = p_i \star p_i = (p_i)^2$$

- convention: $\bar{\circ}$ is evaluated before $\circ$

# Nonassociative quantum mechanics

A *state* $\rho$ is an expression of the form

$$\rho = \sum_{\alpha=1}^{n} \lambda_\alpha \, \psi_\alpha \, \bar{\circ} \, \psi_\alpha^* \qquad \text{with} \qquad \int |\psi_\alpha|^2 = 1$$

$\lambda_\alpha$ are probabilities and $\psi_\alpha$ are *phase space wave functions*:

Expectation value:

$$\langle A \rangle = \int A \star \rho = \sum_\alpha \lambda_\alpha \int \psi_\alpha^* \star (A \star \psi_\alpha) = \int A \cdot S_\rho \ ,$$

with *state function*

$$S_\rho = \sum_\alpha \lambda_\alpha \psi_\alpha \star \psi_\alpha^* \ , \qquad \int S_\rho = 1 \ .$$

# Nonassociative quantum mechanics

Expectation values of observables (= real functions) are real

$$\langle A\rangle^* = \sum_\alpha \lambda_\alpha \int (A \star \psi_\alpha)^* \star \psi_\alpha = \sum_\alpha \lambda_\alpha \int \psi_\alpha^* \star (A^* \star \psi_\alpha) = \langle A^*\rangle$$

Expectation value of compositions

$$\begin{aligned}\langle A \circ B \circ \ldots \circ C\rangle &= \int (A \circ B \circ \ldots \circ C) \star \big(\sum_\alpha \lambda_\alpha \psi_\alpha \,\bar{\circ}\, \psi_\alpha^*\big)\\ &= \sum_\alpha \lambda_\alpha \int [A \star (B \star \ldots (C \star \psi_\alpha)] \star \psi_\alpha^*\end{aligned}$$

# Nonassociative quantum mechanics

## Positivity

$$
\begin{aligned}
\langle A^* \circ A \rangle &= \sum_\alpha \lambda_\alpha \int \psi_\alpha^* \star [A^* \star (A \star \psi_\alpha)] = \sum_\alpha \lambda_\alpha \int (\psi_\alpha^* \star A^*) \star (A \star \psi_\alpha) \\
&= \sum_\alpha \lambda_\alpha \int (A \star \psi_\alpha)^* \cdot (A \star \psi_\alpha) = \sum_\alpha \lambda_\alpha \int |A \star \psi_\alpha|^2 \geq 0
\end{aligned}
$$

$\leadsto$ semi-definite, sesquilinear form

$$
(A, B) := \langle A^* \circ B \rangle = \sum_\alpha \lambda_\alpha \int (A \star \psi_\alpha)^* \cdot (B \star \psi_\alpha)
$$

$\Rightarrow$ Cauchy-Schwarz inequality

$$
|(A, B)|^2 \leq (A, A)(B, B) \ .
$$

$\leadsto$ uncertainty relations

# Nonassociative quantum mechanics

## Uncertainty relations

uncertainty in terms of shifted coordinates $\widetilde{X}^I = X^I - \langle X^I \rangle$

$$(\Delta X^I)^2 = \langle (X^I)^{\star 2} \rangle - \langle X^I \rangle^2 = \langle \widetilde{X}^I \star \widetilde{X}^I \rangle = \langle \widetilde{X}^I \circ \widetilde{X}^I \rangle = (\widetilde{X}^I, \widetilde{X}^I)$$

Cauchy-Schwarz

$$(\Delta X^I)^2 (\Delta X^J)^2 \geq |(\widetilde{X}^I, \widetilde{X}^J)|^2 = \frac{1}{4} |\langle [X^I, X^J]_\circ \rangle|^2 + \frac{1}{4} |\langle \{\widetilde{X}^I, \widetilde{X}^J\}_\circ \rangle|^2$$

$\Rightarrow$ Born-Jordan-Heisenberg-type uncertainty relation

$$\Delta X^I \cdot \Delta X^J \geq \frac{1}{2} \big| \langle [X^I, X^J]_\circ \rangle \big|$$

# Nonassociative quantum mechanics

## Position-momentum uncertainty

$[p_i, p_j]_\circ = [p_i, p_j]_\star = 0$ and $[p_i, x^j]_\circ = [p_i, x^j]_\star = i\hbar\delta_i^j$ and therefore

$$\Delta p_i \cdot \Delta p_j \geq 0 \qquad \text{and} \qquad \Delta x^i \cdot \Delta p_j \geq \frac{\hbar}{2}\delta_j^i$$

## Position-position uncertainty

$$\begin{aligned}[x^i, x^j]_\circ \star \psi \equiv x^i \star (x^j \star \psi) - x^j \star (x^i \star \psi) &= [x^i, x^j]_\star \star \psi - \hbar^2 R^{ijk} \partial_k \psi \\ &= i\hbar R^{ijk}\big(p_k \psi - \frac{i\hbar}{2}\partial_k \psi + i\hbar \partial_k \psi\big) = i\hbar R^{ijk} \psi \star p_k\end{aligned}$$

and therefore

$$\Delta x^i \cdot \Delta x^j \geq \frac{\hbar}{2}\big| R^{ijk} \langle p_k \rangle' \big| \ ,$$

featuring the opposite (!) state $\rho' = \sum_{\alpha=1}^n \lambda_\alpha \, \psi_\alpha^* \,\bar{\circ}\, \psi_\alpha$

# Nonassociative quantum mechanics

## Eigenfunctions and eigenstates

"star-genvalue equation"

$$A \star f = \lambda f \qquad \text{with } \lambda \in \mathbb{C}$$

complex conjugation implies $f^* \star A^* = \lambda^* f^*$

- real functions have real eigenvalues

$$f^* \star (A \star f) - (f^* \star A) \star f = (\lambda - \lambda^*)(f^* \star f)$$

$$(\lambda - \lambda^*) \int f^* \star f = (\lambda - \lambda^*) \int |f|^2 = 0 \ .$$

- eigenfunctions with different eigenvalues are orthogonal

# Nonassociative quantum mechanics

Associator and common eigen states
if $X^I \star S = \lambda^I S$ and $X^J \star S = \lambda^J S$ and $X^K \star S = \lambda^K S$ then

$$\int [(X^I \star X^J) \star X^K] \star S = \int (X^I \star X^J) \star (X^K \star S)$$
$$= \lambda^K \int (X^I \star X^J) \star S = \lambda^K \int X^I \star (X^J \star S) = \lambda^K \lambda^J \lambda^I$$

likewise $\int [X^I \star (X^J \star X^K)] \star S = \lambda^I \lambda^K \lambda^J$.

taking the difference implies

$$[[X^I, X^J, X^K]]_\star = \lambda^K \lambda^J \lambda^I - \lambda^I \lambda^K \lambda^J = 0$$

$\Rightarrow$ Nonassociating observables do not have common eigen states
$\rightsquigarrow$ spacetime coarse graining

# Nonassociative quantum mechanics

Area and volume operators

$$iA^{IJ} = [\widetilde{X}^I, \widetilde{X}^J]_\star \ \text{ and } \ V^{IJK} = \frac{1}{2}[[\widetilde{X}^I, \widetilde{X}^J, \widetilde{X}^K]]_\star$$

expectation values of these (oriented) area and volume operators:

$$\langle A^{IJ} \rangle = \hbar \Theta^{IJ}(\langle p \rangle) \ \text{ and } \ \langle V^{IJK} \rangle = \frac{3}{2}\hbar^2 R^{IJK}$$

with three interesting special cases

$$\langle A^{(x^i, p_j)} \rangle = \hbar \delta^i_j \ , \quad \langle A^{ij} \rangle = \hbar R^{ijk} \langle p_k \rangle \ , \quad \langle V^{ijk} \rangle = \frac{3}{2}\hbar^2 R^{ijk}$$

$\Rightarrow$ coarse-grained spacetime with quantum of volume $\frac{3}{2}\hbar^2 R^{ijk}$

# Nonassociative quantum mechanics

$$\rho = \sum_{\alpha=1}^{n} \lambda_\alpha \, \psi_\alpha \, \bar{\circ} \, \psi_\alpha^* \ , \qquad S_\rho = \sum_\alpha \lambda_\alpha \psi_\alpha \star \psi_\alpha^* \ , \qquad \mathcal{H} \in \mathbb{R}$$

Evolution (Schrödinger-style):

$$i\hbar \frac{\partial \psi}{\partial t} = \mathcal{H} \star \psi \ , \qquad \mathcal{H} \star \psi = E \, \psi$$

$$\frac{\partial A}{\partial t} = \frac{i}{\hbar} \, [\mathcal{H}, A]_\circ \qquad (\circ\text{-derivation})$$

Evolution (Heisenberg-style):

$$\frac{\partial A}{\partial t} = \frac{i}{\hbar} \, [\mathcal{H}, A]_\star \qquad (\text{not a } \star\text{-derivation!})$$

$$\frac{\partial S_\rho}{\partial t} = \frac{1}{i\hbar} \, [\mathcal{H}, S_\rho]_\star \ , \qquad \mathcal{H} \star S_\rho = E \, S_\rho$$

# Remark on Nambu-Poisson 3-brackets

## Nambu-Poisson structures

- Appear in effective membrane actions
- Nambu mechanics: multi-Hamiltonian dynamics with generalized Poisson brackets; e.g. Euler's equations for the spinning top :

$$\frac{d}{dt}L_i = \{L_i, \frac{\vec{L}^2}{2}, T\} \qquad \text{with} \quad \{f, g, h\} \propto \epsilon^{ijk}\, \partial_i f\, \partial_j g\, \partial_k h$$

- more generally

$$\begin{aligned} \{\{f_0, \cdots, f_p\}, h_1, \cdots, h_p\} \quad &= \quad \{\{f_0, h_1, \cdots, h_p\}, f_1, \cdots, f_p\} + \dots \\ &\quad \dots + \{f_0, \dots, f_{p-1}, \{f_p, h_1, \cdots, h_p\}\} \end{aligned}$$

- The nonassociative $\star$-product quantizes these brackets:

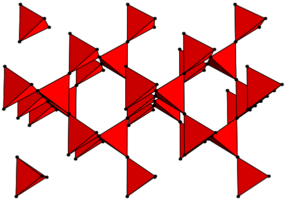
$$\underbrace{[[x^i, x^j, x^k]]_\star}_{\text{Jacobiator}} = i\hbar \sum_l \big(R^{ijl}[p_l, x^k]_\star + \text{ cycl.}\big) = 3\hbar^2 R^{ijk}$$

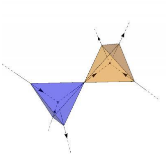
# Summary

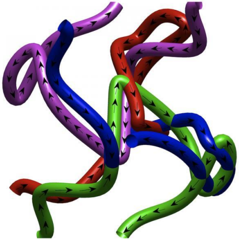
- interaction via deformation $\rightsquigarrow$ generalizes gauge principle
- (non-geometric) fluxes $\rightsquigarrow$ nonassociative structures
- nonassociative quantum mechanics $\rightsquigarrow$ can be formulated
- spacetime (energy-momentum) coarse graining

# Magnetic monopoles in the lab

spin ice pyrochlore and Dirac monopoles







Castelnovo, Moessner, Sondhi (2008)
Fennell; Morris; Hall, . . . (2009)

frustrated spin system $\leftrightarrow$ huge degeneracy of classical ground state

frustration is lifted but pyrochlore spin ice property survives quantization
Lieb, PS (1999)

Thanks for listening!