Curving the Doubled Space: A para-Hermitian geometry for DFT

Recent Advances in T/U-dualities and Generalized Geometries

Ruđer Bošković Institute, Zagreb

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Felix J. Rudolph

Ludwig-Maximilian-Universität München

based on work with
with L. Freidel and D. Svoboda

Key Points – "Curving the Doubled Space"

- Alternative implementation of constraint
- para-Kähler and para-Hermitian geometries
- Connections for these geometries

Other and related Approaches

In the literature

- DFT on group manifolds [Blumenhagen,Haßler,Lüst]
- DFT and QP-manifolds [Heller,Ikeda,Watamura]
- Extended Riemannian Geometry and NQ-manifolds [Deser,Sämann]

Outline

Overview of GG and DFT

Generalized Geometry

[Hitchin, Gualtieri]

Extended vector bundle $E = TM \oplus T^*M$

- Generalized vectors: $X = (x, \alpha) \in \Gamma(E)$
- $O(D, D)$ metric: $\eta(X, Y) = \iota_x \beta + \iota_y \alpha$
- anchor map: $\pi : E \longrightarrow TM, \pi(X) \mapsto x$

Derivative and Bracket

- Dorfman: $\mathbb{L}_X Y = ([x, y], \mathcal{L}_x \beta - \iota_y \mathrm{d}\alpha)$ with $J = 0$
- Courant: $[\![X, Y]\!] = \mathbb{L}_X Y - \frac{1}{2}\mathrm{d}\eta(X, Y)$
- compatible: $\mathbb{L}_X \eta = 0$ and $\pi(\mathbb{L}_X Y) = \mathcal{L}_x y$

Double Field Theory

[Siegel; Hull,Zwiebach]

Double underlying space $\mathcal{P} \cong M \times \tilde{M}$

- Doubled coordinates: $x^M = (x^\mu, \tilde{x}_\mu)$
- Generalized vectors: $X = X^\mu(\frac{\partial}{\partial x^\mu} + \mathrm{d}\tilde{x}_\mu) + X_\mu(\frac{\partial}{\partial \tilde{x}_\mu} + \mathrm{d}x^\mu)$

Generalized Lie derivative

- $\mathbf{L}_X Y^M = X^N \partial_N Y^M - Y^N \partial_N X^M + \eta^{MN} \eta_{PQ} \partial_N X^P Y^Q$
- Algebra does not close: $J \neq 0$
- Need constraint: $\eta^{AB} \partial_A \partial_B = 0$

GG vs DFT

Generalized Geometry

- No need for section condition
- Basespace is fixed - only $O(D, D)$ transformations on E

Double Field Theory

- T-duality changes basespace: map from M to $\tilde{M}$
- Section condition required

GG vs DFT

Relation between GG and DFT

- After imposing section condition: only depend on half the coordinates $\rightarrow$ DFT reduces to GG
- But different ways of picking spacetime M: which half of $\mathcal{P}$ is base for GG?

Need extra geometric information to relate GG to DFT

Restricted Fields vs. Restricted Derivatives

Different Approach

Consider the metric algebroid $(E, \eta, \pi, \mathbf{L}^{\nabla})$

- Vector bundle $E = T\mathcal{P}$
- metric η and compatible connection ∇
- anchor $\pi : E \to T\mathcal{P}$
- generalized Lie derivative

$$\mathbf{L}^{\nabla}_X Y = \nabla_{\pi(X)} Y - \nabla_{\pi(Y)} X + \theta_{\nabla}(Y, X)$$

"Twist" Vector θ_{∇}

- Given by $\eta(Z, \theta_{\nabla}(Y, X)) = \eta(Y, \nabla_{\pi(Z)} X)$
- Like Y-tensor: $\theta^A_{\nabla}(Y, X) = Y^{AB}{}_{CD}\, \pi^E{}_B \nabla_E X^C Y^D$

Different Approach

For $\nabla = \partial$ and $\pi = \mathrm{Id}$ this is the algebroid for DFT

- C-bracket: $[\![X,Y]\!] = \mathbf{L}^{\partial}_{X} Y - \frac{1}{2}\mathrm{d}\eta(X,Y)$
- Jacobiator: $J = \eta(\theta,\theta)$

To get Jacobi identity $\Rightarrow$ need section condition:

$$\eta_{AB}\theta^{A}\theta^{B} = 0$$

Now try different approach:
instead of restricting the fields, restrict the derivatives!

Projections & Derivatives

Projection operators on $T\mathcal{P} = L \oplus \tilde{L}$

- $P, \tilde{P} : T\mathcal{P} \to T\mathcal{P}$
- Maximally isotropic w.r.t. η: $\quad L = \operatorname{Im} P, \quad \tilde{L} = \operatorname{Im} \tilde{P}$
- Important property: $\quad \eta(P(X), Y) = \eta(X, \tilde{P}(Y))$

Projected derivative

- For any metric compatible connection ∇: $\quad \mathrm{D}_X := \nabla_{P(X)}$
- Projected generalized Lie derivative

$$\mathbf{L}^{\mathrm{D}}_X Y = \mathrm{D}_X Y - \mathrm{D}_Y X + \theta_{\mathrm{D}}(Y, X)$$

Vaisman Formalism

[Vaisman '12]

For $\nabla = \partial$ and $\pi = P$ can show

- since $\theta_{\mathrm{D}} \in \tilde{L} = \mathrm{Ker}\, P$, i.e. $P(\theta_{\mathrm{D}}) = 0$
- have $\eta(\theta_{\mathrm{D}}, \theta_{\mathrm{D}}) = 0$
- and $J^{\mathrm{D}} = 0$

Thus $(E = T\mathcal{P}, \eta, \pi = P, \mathbf{L}^{\mathrm{D}})$ is a Leibniz algebroid

- The Jacobi identity holds
- The fields remain unrestricted

Restricted Fields vs. Restricted Derivatives

- Different but equivalent approach to close algebra
- Fields remain unrestricted, but $\mathbf{L}^{\mathrm{D}}$ only "sees" half the coordinate dependence
- Different solutions to section condition $\Leftrightarrow$ different splittings of $T\mathcal{P}$

para-Hermitian Geometry

Doubled space = symplectic manifold

- extra ingredient: symplectic form ω [Hull '04; Deser,Sämann '16]
- almost para-Hermitian manifold $\mathcal{P}$ with (η, ω) and $K := \eta^{-1}\omega$
- bi-Lagrangian structure K: $T\mathcal{P} = L \oplus \tilde{L}$

$$K\big|_L = +1, \quad K\big|_{\tilde{L}} = -1 \quad \text{with} \quad K^2 = +1$$

Two aims:

- find general / canonical connections in para-Hermitian geometry
- show how ω provides the extra geometrical information to relate GG and DFT

para-Kähler and para-Hermitian Geometries

Almost Para-Hermitian Geometry

Manifold with three compatible structures

- Almost para-complex structure K: $K^2 = +1$
- Almost symplectic structure ω: $K^{\mathsf{T}} \omega K = -\omega$
- pseudo-Riemannian structure η: $\eta(X, Y) = \omega(X, K(Y))$

Almost Para-Kähler Geometry

Integrability conditions

- If ω is closed: $\mathrm{d}\omega = 0 \Rightarrow$ Almost Para-Kähler
- If K is integrable: $N_K = 0 \Rightarrow$ Para-Hermitian
- If K is integrable and ω is closed: $\Rightarrow$ Para-Kähler

Geometry for DFT

Bi-Lagrangian manifold $\mathcal{P}$

- metric algebroid structure on $T\mathcal{P} = L \oplus \tilde{L}$
- C-bracket and generalized Lie derivative
- fluxes appear in $\mathrm{d}\omega \Rightarrow$ para-Kähler or para-Hermitian

Jacobi identity

Define Jacobiator

$$J^D(X,Y,Z,W) := \eta([\mathbf{L}^{\mathrm{D}}_X, \mathbf{L}^{\mathrm{D}}_Y]Z - \mathbf{L}^{\mathrm{D}}_{\mathbf{L}^{\mathrm{D}}_X Y} Z, W)$$

Define projected tensors

- curvature: $R_{\mathrm{P}}(X,Y)Z := [\mathrm{D}_X, \mathrm{D}_Y]Z - \mathrm{D}_{[P(X),P(Y)]}Z$
- torsion: $\tau_{\mathrm{P}}(X,Y) := P([\mathrm{D}_X Y - \mathrm{D}_Y X] - [P(X),P(Y)])$
- Nijenhuis: $N_{\mathrm{P}}(X,Y) := \tilde{P}([P(X),P(Y)])$

Related to

- Usual curvature:

$$R_{\mathrm{P}}(X,Y)Z = R(P(X),P(Y))Z + \nabla_{N_{\mathrm{P}}(X,Y)}Z$$

- Usual torsion:

$$\begin{aligned}\tau_{\mathrm{P}}(X,Y) = T(P(X),P(Y)) + N_{\mathrm{P}}(X,Y) \\ - \frac{1}{2}\big[(D_X K)Y - (D_Y K)X\big]\end{aligned}$$

Jacobi identity

$$
\begin{aligned}
J^{\mathrm{D}}(X,Y,Z,W) = {} & \\
& R_{\mathrm{P}}(X,Y,Z,W) + R_{\mathrm{P}}(Y,Z,X,W) + R_{\mathrm{P}}(Z,X,Y,W) \\
& \quad - R_{\mathrm{P}}(W,Z,X,Y) - R_{\mathrm{P}}(W,X,Y,Z) - R_{\mathrm{P}}(W,Y,Z,X) \\
& - \eta(W, \nabla_{\tau_{\mathrm{P}}(X,Y)} Z) - \eta(W, \nabla_{\tau_{\mathrm{P}}(Y,Z)} X) - \eta(W, \nabla_{\tau_{\mathrm{P}}(Z,X)} Y) \\
& \quad - \eta(Z, \nabla_{\tau_{\mathrm{P}}(X,W)} Y) - \eta(X, \nabla_{\tau_{\mathrm{P}}(Y,W)} Z) + \eta(Y, \nabla_{\tau_{\mathrm{P}}(W,Z)} X) \\
& + \eta(\theta_{\mathrm{D}}(Z,X), \theta_{\mathrm{D}}(W,Y)) - \eta(\theta_{\mathrm{D}}(Z,Y), \theta_{\mathrm{D}}(W,X)) \\
& \quad - \eta(\theta_{\mathrm{D}}(Y,X), \theta_{\mathrm{D}}(W,Z))
\end{aligned}
$$

Jacobi identity

Three contributions: curvature, torsion, twist θ_{D}

- $L, \tilde{L}$ Lagrangian $\Rightarrow P(\theta_{\mathrm{D}}) = 0 \Rightarrow \eta(\theta_{\mathrm{D}}, \theta_{\mathrm{D}}) = 0$
- Integrability of L = anchoring of $P \Rightarrow \tau_{\mathrm{P}} = 0$
- Check R_{P} individually

Canonical Connections

Para-Kähler

Levi-Civita connection $\mathring{\nabla}$ of η

- symplectic form is closed: $\mathrm{d}\omega = 0$
- bi-Lagrangian structure is integrable: $N_K = 0$
- then $\mathring{\nabla}\eta = 0$ and $\mathring{\nabla}\omega = 0$

Generalized Lie derivative

- projected derivative $\mathring{\mathrm{D}}_X = \mathring{\nabla}_{P(X)}$
- generalized Lie derivative $\mathbf{L}^{\mathring{\mathrm{D}}}_X$ with $J^{\mathring{\mathrm{D}}} = 0$
- since $\mathring{R}_{\mathrm{P}} = 0$ and $\mathring{\tau}_{\mathrm{P}} = 0$

Para-Kähler

For DFT on para-Kähler manifold $(\mathcal{P}, \eta, \omega)$:
$\mathbf{L}^{\mathring{\mathrm{D}}}_X$ is a generalized Lie derivative that satisfies the Jacobi identity for arbitrary metric η

But does not work if $\mathrm{d}\omega \neq 0$ — can we do better?

Para-Hermitian

Canonical para-Hermitian connection ∇^c

- relax closure condition: $\mathrm{d}\omega \neq 0$
- but still integrable: $N_K = 0$
- introduce

$$\nabla^c_X = P\mathring{\nabla}_X P + \tilde{P}\mathring{\nabla}_X \tilde{P}$$

- given by contorsion

$$\eta(\nabla^c_X Y, Z) = \eta(\mathring{\nabla}_X Y, Z) - \frac{1}{2}\mathring{\nabla}_X \omega(Y, K(Z))$$

Para-Hermitian

Properties

- compatible with η and ω: $\nabla^c\eta = 0$ and $\nabla^c\omega = 0$
- compatible with K: $\nabla^c K = K\nabla^c$
- generalized torsion

$$\mathcal{T}^c(X,Y,Z) = \frac{1}{2}\mathrm{d}\omega(K(X),K(Y),K(Z)) - \frac{1}{4}\sum_{cycl.} N_K(X,Y,Z)$$

defined as
$\mathcal{T}(X,Y,Z) := \eta(\mathbf{L}^{\nabla}_X Y - \mathbf{L}^{\mathring{\nabla}}_X Y, Z) = \sum_{cycl.} \Omega(X,Y,Z)$

Para-Hermitian

Generalized Lie derivative

- projected derivative $\mathrm{D}^c_X = \nabla^c_{P(X)}$
- generalized Lie derivative $\mathbf{L}^{\mathrm{D}^c}_X$ with $J^{\mathrm{D}^c} = 0$
- since R^c_{P} terms in J^c vanish by Bianchi identity for $\mathring{R}$
- projected torsion: $\tau^c_{\mathrm{P}} = \mathring{\tau}_{\mathrm{P}} = 0$

Para-Hermitian

For DFT on para-Hermitian manifold $(\mathcal{P}, \eta, \omega)$:
$\mathbf{L}_X^{\mathrm{D}^c}$ is a generalized Lie derivative that satisfies the Jacobi identity for arbitrary metric η

Relation to Generalized Geometry

Relation to Generalized Geometry

Invertible map $\rho : TM \oplus T^*M \to T\mathcal{P}|_M$

$$\rho : (x, \alpha) \mapsto x + \eta^{-1}(\alpha) := x + \tilde{x} = X$$
$$\rho^{-1} : X \mapsto \big(P(X), \eta(\tilde{P}(X))\big)$$

where M is a leaf of the foliation $\mathcal{F}$ of $\mathcal{P}$ $\to$ ρ can be extended to all of $T\mathcal{P}$

Relation to Generalized Geometry

Translate $\mathbb{L}_{(x,\alpha)} = ([x,y], \mathcal{L}_x\beta - \iota_y \mathrm{d}\alpha)$ to some $\mathbf{L}_X Y$

$$\mathbb{L}_{\rho(x,\alpha)}\rho(y,\beta) = [x,y] + \eta^{-1}(\mathcal{L}_x\beta - \iota_y \mathrm{d}\alpha) = \mathbf{L}_X Y$$

Can rewrite this as (using $x = P(X)$, $\tilde{x} = \tilde{P}(X)$, etc.)

$$\eta(\mathbf{L}_X Y, Z) = \eta(\nabla^c_{P(X)} Y - \nabla^c_{P(Y)} X, Z) + \eta(\nabla^c_{P(Z)} X, Y)$$

$$\boxed{\mathbf{L}_X^{\mathrm{D}^c} Y = \mathbb{L}_{\rho(x,\alpha)}\rho(y,\beta)}$$

Relation to Generalized Geometry

Presence of symplectic structure ω gives map ρ

- ρ maps $\mathbb{L}_{(x,\alpha)}$ of GG to generalized Lie derivative $\mathbf{L}_X^{\mathrm{D}^c}$ of Leibniz algebroid $(T\mathcal{P}, \eta, P, \mathbf{L}^{\mathrm{D}^c})$
- Yields canonical connection ∇^c
- Alternative proof of Jacobi identity for $\mathbf{L}_X^{\mathrm{D}^c}$
- Specifies relation between GG and DFT:
  spacetime M is leaf of foliation $\mathcal{F}$ with tangent $L = \operatorname{Im} P$

Outlook

- Include generalized metric $\mathcal{H}$
- Relate to "Born connection" (compatible with $(\eta, \omega, \mathcal{H})$)
- Relax integrability constraints

Summary

- Close algebra by using projected derivatives instead of section condition
- DFT lives on a para-Hermitian manifold – includes a symplectic structure ω
- Canonical connection and its generalized Lie derivative satisfy Jacobi identity

String Theory

Tseytlin Action on phase space with $X = (x/\lambda, y/\epsilon)$

$$S = \frac{1}{2} \int \mathrm{d}\tau \mathrm{d}\sigma \left[(\eta_{AB} + \omega_{AB}) \partial_\tau X^A \partial_\sigma X^B - \mathcal{H}_{AB} \partial_\sigma X^A \partial_\sigma X^B \right]$$

- including topological term [Giveon, Rocek; Hull]

String Theory

- chiral structure: $J = \eta^{-1} \mathcal{H}$
- T-duality on target space: $X \to J(X)$
- ω and K not required, but present

Para-quaternionic Manifold

Born Geometry $(\mathcal{P}; \eta, \omega, \mathcal{H}) \longrightarrow$ para-quaternions (I, J, K)

- complex structure $I = \mathcal{H}^{-1}\omega \qquad (I^2 = -1)$
- chiral structure $J = \eta^{-1}\mathcal{H} \qquad (J^2 = +1)$
- real structure $K = \eta^{-1}\omega \qquad (K^2 = +1)$

All mutually anti-commute

$$\begin{aligned} I &= JK = -KJ, \\ J &= IK = -KI, \\ K &= JI = -IJ, \end{aligned} \qquad\qquad IJK = -1$$

The Nijenhuis Tensor

The Nijenhuis Tensor of a tangent bundle structure A

$$N_A \in \Gamma(\Lambda^2(\mathcal{P}) \otimes \mathfrak{X}(\mathcal{P}))$$

$$N_A(X,Y) = A\big([A(X),Y]+[X,A(Y)]\big) - [A(X),A(Y)] - A^2[X,Y]$$

If A is integrable, $N_A = 0$.

The Nijenhuis Tensor

For bi-Lagrangian structure $K = \eta^{-1}\omega$

$$
\begin{aligned}
N_K(X,Y,Z) = {} & \mathring{\nabla}_Y\omega(X,K(Z)) - \mathring{\nabla}_X\omega(Y,K(Z)) \\
& + \mathring{\nabla}_{K(Y)}\omega(X,Z) - \mathring{\nabla}_{K(X)}\omega(Y,Z) \\
= {} & \mathrm{d}\omega(K(X),K(Y),K(Z)) + \mathrm{d}\omega(X,Y,K(Z)) \\
& + 2\mathring{\nabla}_{K(Z)}\omega(X,Y)
\end{aligned}
$$