Comments on non-isometric T-duality

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Based on [1705.09254]
with P. Bouwknegt, C. Klimčík, and K. Wright

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Outline

1. Isometric T-duality
2. Non-isometric T-duality
3. Equivalence
4. Examples
Consider a non-linear sigma model \( X : \Sigma \rightarrow M \) described by the following action:

\[
S = \int_{\Sigma} g_{ij} \, dX^i \wedge \star dX^j + \int_{\Sigma} B_{ij} \, dX^i \wedge dX^j
\]

In this talk we will ignore the dilaton, and assume that both \( g \) and \( B \) are globally defined fields on \( M \).
Suppose now that there are vector fields generating the following global symmetry:

\[ \delta_\epsilon X^i = v^i_a \epsilon^a \]

for \( \epsilon^a \) constant. The sigma model action is invariant under this transformation if

\[ \mathcal{L}_{v_a} g = 0 \quad \mathcal{L}_{v_a} B = 0 \]

If this is the case, we can gauge the model by promoting the global symmetry to a local one.
The gauged action

Introducing gauge fields $A^a$ and Lagrange multipliers $\eta_a$, the gauged action is

$$S_G = \int_\Sigma g_{ij} DX^i \wedge \star DX^j + \int_\Sigma B_{ij} DX^i \wedge DX^j + \int_\Sigma \eta_a F^a$$

where

- $F = dA + A \wedge A$ is the standard Yang-Mills field strength
- $DX^i = dX^i - \nu^i_a A^a$ are the gauge covariant derivatives.
Gauge invariance

The gauged action is invariant with respect to the following (local) gauge transformations:

\[ \delta_\epsilon X^i = v^i_a \epsilon^a \]
\[ \delta_\epsilon A^a = d\epsilon^a + C^a_{bc} A^b \epsilon^c \]
\[ \delta_\epsilon \eta_a = -C^c_{ab} \epsilon^b \eta_c \]
T-duality

\[ S_G[X, A, \eta] \]

\[ S[X] \]

\text{gauge isometries}
Varying the Lagrange multipliers forces the field strength $F$ to vanish. If we then fix the gauge $A = 0$ we recover the original model.
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On the other hand, we can eliminate the non-dynamical gauge fields $A$, obtaining the dual sigma model.
Can we do it without isometries?

The existence of global symmetries is a very stringent requirement. A generic metric will not have any Killing vectors.

Question

Is it possible to follow the same procedure when the vector fields are not Killing vectors?
Kotov and Strobl\textsuperscript{1} introduced a method of gauging a sigma model without requiring the model to possess isometries.

Their method uses Lie algebroids, and generalises the standard gauging in two notable ways:

- The structure constants of the Lie algebra are promoted to structure functions:
  \[ [v_a, v_b] = C_{ab}^c(X) v_c \]

- The gauge invariance of the gauged action doesn't require the original vector fields to be isometries:
  \[ \mathcal{L}_{v_a} g \neq 0 \quad \mathcal{L}_{v_a} B \neq 0 \]

\textsuperscript{1}[1403.8119]
Chatzistavrakidis, Deser, and Jonke\textsuperscript{2} applied this non-isometric gauging to the Buscher procedure we just reviewed.

They promote the structure constants to functions, and introduce a matrix-valued one-form $\omega^b_a$ satisfying

\[
\mathcal{L}_{v_a} g = \omega^b_a \vee \iota_{v_b} g
\]

\[
\mathcal{L}_{v_a} B = \omega^b_a \wedge \iota_{v_b} B
\]

\textsuperscript{2}[1509.01829] and [1604.03739]
The gauged action

The gauged action is almost the same:

\[ S^\omega_G = \int_\Sigma g_{ij} DX^i \wedge \star DX^j + \int_\Sigma B_{ij} DX^i \wedge DX^j + \int_\Sigma \eta_a F^a_\omega \]

where the curvature is now given by

\[ F^a_\omega = dA^a + \frac{1}{2} C^a_{bc}(X) A^b \wedge A^c - \omega^a_{bi} A^b \wedge DX^i \]
The modified gauge transformations are now

\[
\delta_\epsilon X^i = \nu^i_a \epsilon^a \\
\delta_\epsilon A^a = d\epsilon^a + C^a_{bc} A^b \epsilon^c + \omega^a_{bi} \epsilon^b D X^i \\
\delta_\epsilon \eta_a = -C^c_{ab} \epsilon^b \eta_c + \nu^i_a \omega^c_{bi} \epsilon^b \eta_c
\]
T-duality

$S^\omega_G[X, A, \eta]$

exotically gauge

$S[X]$
T-duality

\[ S_G^{\omega}[X, A, \eta] \]

Integrate and fix gauge exotically gauge

Isometric T-duality

Non-isometric T-duality

Equivalence

Examples
As with isometric T-duality, we can integrate out the fields in two different ways, obtaining the original model or a dual model. In principle, we could use this to construct T-duals of spaces which have no isometries.
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In principle, we could use this to construct T-duals of spaces which have no isometries.
This proposal is equivalent to non-abelian T-duality.\(^3\)

That is, if we can find a set of vector fields and \(\omega_a^b\) which give a non-isometric T-dual, then there exists a set of Killing vectors for the model. The T-dual with respect to these Killing vectors is the same as the non-isometric T-dual.

\(^3\)[1705:09254] P. Bouwknegt, M.B., C. Klimčík, K. Wright
A necessary condition for gauge invariance

Gauge invariance of the action requires the structure functions to be constant, as well as the vanishing of the following variation:

$$\delta_\epsilon(\eta_a F^a_\omega) = \eta_a (d\omega^a_b + \omega^a_c \wedge \omega^c_b) \epsilon^b + O(A) + O(A^2).$$

We therefore require that $\omega^b_a$ is flat:

$$R^b_a = d\omega^b_a + \omega^b_c \wedge \omega^c_a = 0,$$

and this tells us that $\omega^b_a$ is of the form $K^{-1} dK$ for some $K^b_a(X)$. 
A field redefinition

Using this $K$, we can perform the following field redefinitions:

\[ \hat{A}^a = K^a_b A^b \]
\[ \hat{\eta}^a = \eta_b (K^{-1})^b_a \]
\[ \hat{\nu}_a = \nu^i_b (K^{-1})^b_a \]
The gauged action can now be rewritten in terms of the new fields \((X^i, \hat{A}^a, \hat{\eta}_a)\).

\[
S_G^\omega[X, \hat{A}, \hat{\eta}] = \int_\Sigma g_{ij} \hat{D}X^i \wedge \star \hat{D}X^j + \int_\Sigma B_{ij} \hat{D}X^i \wedge \hat{D}X^j + \int_\Sigma \hat{\eta}_a \hat{F}^a = S_G[X, \hat{A}, \hat{\eta}]
\]

where

\[
\hat{F}^a = d\hat{A}^a + \frac{1}{2} \hat{C}^a_{bc} \hat{A}^b \wedge \hat{A}^c
\]
The gauge transformations become the usual non-abelian gauge transformations, and a short computation reveals

$$L_{\hat{\nabla}a} g = 0 \quad L_{\hat{\nabla}a} B = 0$$

**Conclusion**

This proposal is equivalent, via a field redefinition, to the standard non-abelian T-duality
First example

Consider the 3D Heisenberg Nilmanifold, or twisted torus. It has a metric given by

\[
    ds^2 = dx^2 + (dy - x
dz)^2 + dz^2
\]

The non-abelian T-dual of this space is given by

\[
    \hat{ds}^2 = dY^2 + \frac{1}{1 + Y^2} \left( dX^2 + dZ^2 \right)
\]

\[
    \hat{B} = \frac{Y}{1 + Y^2} dX \wedge dZ
\]
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First example

We can gain a better understanding of the geometry by writing the manifold as a group:

\[ \text{Heis} := \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\} \]

(left-invariant) MC forms = \((dx, dy - xdz, dz)\)
(right-invariant) vector fields = \((\partial_x + z\partial_y, \partial_y, \partial_z)\)
First example

We could instead try to gauge this space non-isometrically using the left-invariant vector fields: \( \{ \partial_x, \partial_y, x\partial_y + \partial_z \} \).

These are not all isometries:

\[
\mathcal{L}_{v_1} g = -dy \otimes dx - dz \otimes dy + 2xdz \otimes dz
\]
\[
\mathcal{L}_{v_2} g = 0
\]
\[
\mathcal{L}_{v_3} g = dx \otimes dy + dy \otimes dx - xdx \otimes dz - xdz \otimes dx
\]

and they don’t commute:

\[
[v_1, v_3] = v_2,
\]

however...
First example\textsuperscript{4}

If we take $\omega_3^2 = dx$ and $\omega_1^2 = -dz$, with other components vanishing, the non-isometric gauging constraints are satisfied and we can calculate the non-isometric T-dual model.

\[
\hat{ds}^2 = dY^2 + \frac{1}{1 + Y^2} \left( dX^2 + dZ^2 \right) \\
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\textsuperscript{4}Gauged non-isometrically in [1509:01829]
A. Chatzistavrakidis, A. Deser, L. Jonke
First example$^4$

If we take $\omega_2^2 = dx$ and $\omega_1^2 = -dz$, with other components vanishing, the non-isometric gauging constraints are satisfied and we can calculate the non-isometric T-dual model.

\[
\hat{\text{d}s}^2 = dY^2 + \frac{1}{1 + Y^2} (dX^2 + dZ^2)
\]

\[
\hat{B} = \frac{Y}{1 + Y^2} dX \wedge dZ
\]

Unsurprisingly, it is also the T-fold.

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$^4$Gauged non-isometrically in [1509:01829]
A. Chatzistavrakidis, A. Deser, L. Jonke
Consider $S^3$ with the round metric and $B = 0$.

This metric has an $SO(4)$ group of isometries, and we can find the non-abelian T-dual with respect to an $SU(2)$ subgroup of this.
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The non-abelian T-dual is well-known. The metric is the ‘cigar’ metric, and there is also a non-zero B-field.
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We can write the round metric as

\[ g = \lambda^1 \otimes \lambda^1 + \lambda^2 \otimes \lambda^2 + \lambda^3 \otimes \lambda^3 \]

where the \( \lambda^i \) are the left-invariant Maurer-Cartan forms.

The right-invariant vector fields are isometries of this metric, so let’s try gauging with respect to the left-invariant vector fields\(^5\).

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\(^5\)These also happen to be isometries of the metric, but let’s try to gauge them non-isometrically
Second example

The Lie derivatives of the metric with respect to the left-invariant vector fields, $L_a$ are

$$\mathcal{L}_{L_a}g = - \sum_b C_{ac}^b \lambda^c \lor \lambda^b$$

$$= - C_{ac}^b \lambda^c \lor \iota_{L_b}g$$

We can do non-isometric T-duality by taking $\omega_a^b = - C_{ac}^b \lambda^c$. 
Second example

The remaining gauging constraints are satisfied, and we can calculate the non-isometric T-dual. It is the 'cigar' metric, as expected.
Comments

- The equivalence of non-isometric and non-abelian T-duality remains valid for non-exact $H$
- Geometric interpretation of $\omega^b_a$ as a connection on a Lie algebroid
- There are proposals for alternate gauging. Unknown how to incorporate into T-duality
  - non-flat $\omega^b_a$
  - include a term $\phi^{b}_{ai} \epsilon^b \star DX^i$ into $\delta_\epsilon A^a$
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Geometric interpretation of $\omega^b_a$ as a connection on a Lie algebroid

There are proposals for alternate gauging. Unknown how to incorporate into T-duality
  - non-flat $\omega^b_a$
  - include a term $\phi^b_{ai} \epsilon^b \star DX^i$ into $\delta_\epsilon A^a$

Thanks!