Generalised Brane Calibrations from Exceptional Sasaki-Einstein Structures

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Recent Advances in T/U dualities and Generalised Geometries - Zagreb

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Summary

- Motivation and inspiration
  - Extended objects in String theory: Branes
  - Calibrations of branes
  - G-structures and backgrounds
- Generalised Geometry
  - Extended bundles
  - Exceptional G-structures
- Calibrations from Exceptional Structures
  - Probes in AdS backgrounds and Exceptional Sasaki-Einstein structures

Branes

- String theory admits extended objects: strings and branes

- Various String theories, various branes

$$\begin{array}{l|cc} \text{Type IIA} & D_p & p \text{ even} \\ \text{Type IIB} & D_p & p \text{ odd} \\ \text{M-Theory} & M_2,\ M_5 & \end{array}$$

- A *Dp* brane carries a charge associated to a (p+1)-form potential $A^{(p+1)}$

- They are sources for these fields

- Given a spacetime, which are the brane configurations at minimal energy?

How to survive this talk

- A supersymmetric background is a solution to the sugra e.o.m. with fermions set to zero and vanishing supersymmetric variations for bosons
  - A fluxless background is a background where all fields except the metric are set to zero
- When a D brane probes space-time, a gauge theory lives on its world volume
  - There are fields living only on the world volume
  - Configurations of branes preserving the supersymmetry of the background are called BPS
  - Calibrations can be used to encode BPS conditions

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- Bonus suggestion: take a nap

Why (and where) studying brane probes?

Compactifications: One looks for solutions of the following form

$$\mathcal{M}_D \to \mathcal{M}_{D-d} \times M_d$$

External space

Internal space

We are going to focus on $\mathcal{N} = 2$ AdS spacetimes and various brane configurations.

These configurations are interesting for various applications in holography

There are restrictions on the geometry of AdS backgrounds

Anti-de Sitter backgrounds with fluxes

[Gauntlett, Martelli, Sparks, Waldram]
[Gabella, Gauntlett, Palti, Sparks, Waldram]

- Our goal is to use the geometrical structures for AdS backgrounds

- Supersymmetry implies the existence of geometrical structures on the internal space.

- **Example**: Calabi-Yau 3-fold
  - Solutions of $M_4 \times CY_6$ with no fluxes
  - Supersymmetry
    - Existence of a never vanishing spinor ϵ
    - Covariantly constant spinor $\nabla\epsilon = 0$

$\mathcal{N} = 2$ sugra on M_4

We can construct on CY_6 an integrable SU(3) structure

$$\{\omega, \Omega\}$$

Integrable structures

- The CY has an integrable SU(3) structure
  - Holomorphic 3-form Ω $\quad \Omega \wedge \bar{\Omega} \neq 0$
  - Real 2-form ω $\quad \omega \wedge \omega \wedge \omega \neq 0$
  - Integrability: closed forms $\quad \mathrm{d}\Omega = \mathrm{d}\omega = 0$
- Defining never vanishing tensors means defining G-structures
  - Ω defines a $SL(3,\mathbb{C})$ structure
  - ω defines a $Sp(3,\mathbb{R})$ structure
  - Compatibility $\omega \wedge \Omega = 0$ $\quad SL(3,\mathbb{C}) \cap Sp(3,\mathbb{R}) = SU(3)$
- How to generalise this for generic fluxes?

Generalised Geometry

[Hitchin, Gualtieri, '01]

- Ordinary geometry is the study of structures on TM
- In GR vectors generate diffeomorphisms via Lie derivative
- One needs a formalism where "vectors" generate symmetries of supergravity

The main idea: define a generalised tangent bundle

$$E \cong TM \oplus T^*M$$

$O(d,d)$

Structure group

generalised vector $\quad V = v + \lambda$

1-form

vector

- The structure group of the generalised tangent bundle is $O(d,d)$ the T-duality group of toroidal compactifications.

How do we insert fluxes?

- *O(d,d)* formalism encodes the 3-form flux $H = \mathrm{d}B$
  - The adjoint action naturally contains a 2-form
  - Define the twisted generalised vector

$$V = e^{-B}\tilde{V} = v + \lambda - \iota_v B$$

adjoint action of O(d,d)

This determines the topology of E

  - Patchings: on an overlapping of patches $U_\alpha \cap U_\beta$

$$V_\alpha = e^{-d\Lambda_{\alpha\beta}} V_\beta \quad \Longleftrightarrow \quad B_{(\alpha)} = B_{(\beta)} - \mathrm{d}\Lambda_{(\alpha\beta)}$$

2-form

Connection on a gerbe
This corresponds to gauge transformations of NSNS supergravity gauge potential.

Exceptional Generalised Geometry

[Hull; Pacheco, Waldram]
[Coimbra, Strickland-Constable, Waldram]

- One wants to include RR fields
  - T-duality group generalises to U-duality: define a generalised tangent bundle with a structure group given by the U-duality one.
- EGG depends on the theory: focus on M-theory on $\mathrm{AdS}_5 \times M_6$
  - Generalised tangent bundle

$$E \cong TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \oplus \cancel{(T^*M \otimes \Lambda^7 T^*M)}$$

$\tilde{V} = (v, \omega, \sigma, \tau)$ generalised vector

charges of wrapped M-branes

Structure group $E_{7(7)}$

- Potentials live in the adjoint bundle

$$\mathrm{ad}\ \tilde{F} \cong \mathbb{R} \oplus (TM \otimes T^*M) \oplus \Lambda^3 TM \oplus \Lambda^3 T^*M \oplus \Lambda^6 TM \oplus \Lambda^6 T^*M$$

$$\mathcal{A} = \Big(\dots, A, \dots, \tilde{A}, \dots\Big)$$

- E has a fibered structure

Adjoint rep

$$V = e^{A+\tilde{A}}\tilde{V} \qquad R = e^{A+\tilde{A}}\tilde{R}\, e^{-A-\tilde{A}}$$

- Ordinary Lie derivative generates diffeomorphisms

$$\mathcal{L}_v w^\mu = v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu = v^\nu \partial_\nu w^\mu - (\partial \otimes_{\mathrm{ad}} v)^\mu{}_\nu\, w^\nu$$

$gl(d, \mathbb{R})$

- Dorfman Derivative *[Pacheco, Waldram]*

$$L_V V' = V \cdot \partial V' - (\partial \otimes_{\mathrm{ad}} V) \cdot V'$$

Generalised G-structures

- How to use these ingredients to get generalised G-structures?
  - Generalised structures parametrise scalars of the AdS sugra

[Ashmore, Petrini, Waldram]
[Graña, Ntokos]

$$\mathcal{M}_v = \frac{E_{7(7)} \times \mathbb{R}^+}{E_{6(2)}} \longrightarrow K \in \mathbf{56} \quad \text{Vector structure}$$

$$\mathcal{M}_{HK} = \frac{E_{7(7)} \times \mathbb{R}^+}{Spin^*(12)} \longrightarrow J_a \in \mathbf{133} \quad \text{Hypermultiplet structure}$$

  - J_a form an $SU(2)$ triplet

$$\mathcal{M}_{QK} = \frac{\mathcal{M}_{HK}}{SU(2) \times \mathbb{R}^+}$$

Exceptional Sasaki-Einstein Structure

[Ashmore, Petrini, Waldram]
[Graña, Ntokos]

- It is defined by two generalised tensors

$$J_a \in \frac{E_{7(7)} \times \mathbb{R}^+}{Spin^*(12)} \qquad K \in \frac{E_{7(7)} \times \mathbb{R}^+}{E_{6(2)}}$$

- Compatibility

$$J_a \cdot K = 0 \qquad Spin^*(12) \cap E_{6(2)} = SU(6)$$

$$q(K,K,K,K) \propto \mathrm{vol} \qquad \mathcal{N} = 2 \text{ Supersymmetry}$$

- Integrability

$$L_K K = 0 \qquad L_K J_a = \epsilon_{abc} \lambda_b J_c$$

quartic invariant

$$\mu_a(V) = \lambda_a \int q(K)^{-1/2} q(V,K,K,K)$$

- They are equivalent to supersymmetry conditions
- μ_a are moment maps for the action of generalised diffeomorphisms

Comments

- We want to use Exceptional Sasaki-Einstein structures to express generalised calibration forms for brane probes
  - We focus on the vector structure
  - Supersymmetric branes wrapped on internal cycles give rise to AdS particles $\longleftrightarrow$ BPS operators
  - The volume of the brane corresponds to the conformal dimension $\Delta(\mathcal{O})$

- We will see how BPS conditions correspond to integrability of the structures

Generalised Calibrations

[Gutowski, Papadopoulos]
[Gauntlett, Martelli, Pakis, Waldram]
[Koerber, Martucci]

- Problem: Conditions on susy branes wrapping cycles in flux compactifications
- Generalised Calibrations: tool to encode BPS conditions for branes
  - A calibration φ for a *n*-submanifold $\mathcal{N}$ of $\mathcal{M}$ is a closed *n*-form such that at any point of $\mathcal{N}$

[Harvey, Lawson]

$\mathcal{N}$ *is said calibrated*

$$P[\varphi]_{\mathcal{N}} = \mathrm{vol}_{\mathcal{N}}$$

  - The manifold $\mathcal{N}$ is volume minimising in its topological class
  - When branes couple to background fluxes, they are associated to energy minimising submanifolds
- We start from κ-symmetry or from supersymmetry algebra and reduce spinor bilinears according to dimensional reduction
- We find that we can express calibrations in term of Exceptional Structures

Generalised Calibrations
M-theory

- Consider M-theory on $\mathrm{AdS} \times M$
- The Majorana Killing spinor can be used to define bilinear forms

$$\mathcal{K}_M = \bar{\epsilon}\Gamma_M \epsilon$$
$$\omega_{MN} = \bar{\epsilon}\Gamma_{MN}\epsilon$$
$$\Sigma_{M_1 \ldots M_5} = \bar{\epsilon}\Gamma_{M_1 \ldots M_5}\epsilon$$

- Consider supersymmetry algebra with central extensions for branes

$$\{Q_\alpha, Q_\beta\} = C\Gamma^M_{\alpha\beta} P_M + C\Gamma^{MN}_{\alpha\beta} Z_{MN} + C\Gamma^{M_1 \ldots M_5}_{\alpha\beta} Z_{M_1 \ldots M_5}$$

$$(Q\epsilon)^2 = \mathcal{K}^M P_M + \omega^{MN} Z_{MN} + \Sigma^{M_1 \ldots M_5} Z_{M_1 \ldots M_5} \geq 0$$

Example: Fluxless Case

- Consider M-theory on $\mathrm{AdS} \times M$ and no fluxes

$$\nabla\epsilon = 0$$

  - Consider an M2 wrapping a 2 cycle for simplicity

$$(Q\epsilon)^2 = \mathcal{K}^M P_M + \omega_{MN} Z^{MN} \geq 0$$

$$Z^{MN} = \mathrm{d}X^M \wedge \mathrm{d}X^N$$

  - We put ourself in the rest frame of the brane: $P_\mu = (\mathcal{H}, 0, \ldots, 0)$

[Cascales, Uranga]

  - We get the inequality

From DBI action

$$\mathcal{H} = \sqrt{-\det P[G]}\ \mathrm{vol}$$

$$\mathcal{H} \geq P[\omega] = \sqrt{-\det P[G]}\ \underbrace{\bar{\epsilon}\Gamma_{\mu\nu}\epsilon}_{\tilde{\omega}}\ \mathrm{d}\sigma^\mu \wedge \mathrm{d}\sigma^\nu$$

  - The closure follows from Killing spinor equations

$$\mathrm{d}\omega = \mathrm{d}\left(\bar{\epsilon}\Gamma\epsilon\right) = 0$$

Example: Fluxless Case

- We have a closed form satisfying

$$\tilde{\omega} \leq \mathrm{vol}$$

  - Supersymmetry is satisfied when the inequality is saturated
  - It minimises the volume form of the wrapped cycle
  - It minimises the energy of the brane in the case of a fluxless background

- We conclude that supersymmetric M2 branes wrap 2-cycles calibrated with respect to the form $\tilde{\omega}$

Generalised Calibrations

M-theory

- Consider M-theory on $\mathrm{AdS} \times M$
- The Majorana Killing spinor can be used to define bilinear forms

$$\mathcal{K}_M = \bar{\epsilon}\Gamma_M\epsilon$$

$$\omega_{MN} = \bar{\epsilon}\Gamma_{MN}\epsilon$$

$$\Sigma_{M_1\ldots M_5} = \bar{\epsilon}\Gamma_{M_1\ldots M_5}\epsilon$$

- A supersymmetric brane has to satisfy

E.g. M5 brane

$$\hat{\Gamma}\epsilon = \epsilon$$

κ-symmetry

$$||\epsilon||^2 L_{DBI}\mathrm{vol}_5 \geq \frac{1}{2}P[\iota_{\mathcal{K}}H] \wedge H + P[\omega] \wedge H + P[\Sigma]$$

[Gabella, Martelli, Passias, Sparks]

Generalised Calibrations

M-theory

$$\|\epsilon\|^2 L_{DBI}\mathrm{vol}_5 \geq \frac{1}{2}P[\iota_{\mathcal{K}}H]\wedge H + P[\omega]\wedge H + P[\Sigma]$$

Center of AdS

- This bound implies a bound on the energy

$$E_{\mathrm{M5}} \geq E_{\mathrm{M5}}^{BPS} = \int_{\mathcal{S}} P[\Sigma] + P[\iota_{\mathcal{K}}\tilde{A}] + P[\omega]\wedge H + \tfrac{1}{2}P[\iota_{\mathcal{K}}H]\wedge(A-2H)$$

- Generalised calibration form

$$\Phi_{\mathrm{M5}} = \Sigma + \iota_{\mathcal{K}}\tilde{A} + \omega\wedge H + \tfrac{1}{2}\iota_{\mathcal{K}}H\wedge(A-2H)$$

- The expression of this form depends on the brane configuration
- We can show this can be seen from Exceptional Structures

Generalised Calibrations
M-theory

- Consider as example $\mathrm{AdS}_5 \times M_6$ *[Gauntlett, Martelli, Sparks, Waldram]*
  - Metric ansatz $\mathrm{d}s^2 = \mathrm{d}s^2_{\mathrm{AdS}} + e^{2\Delta}\mathrm{d}s^2_{M_6}$
  - Spinor decomposition

$$\epsilon = \psi \otimes \chi + \psi^c \otimes \chi^c = \psi \otimes (\chi_1 + \chi_2^*) + \psi^c \otimes (\chi_1 + \chi_2^*)^c$$

  - The internal manifold has a local SU(2)-structure $\{\xi, Y', Z, \ldots\}$
  - Exceptional Sasaki-Einstein Structure $\{K, J_a\}$ *[Ashmore, Petrini, Waldram]*
  - M5 brane wrapping internal cycles: AdS particle and $H = P[A]$
  - Supersymmetry equations are equivalent to integrability conditions

$$\begin{aligned} \mathrm{d}\left(e^{\Delta} Y'\right) &= -\iota_\xi F \\ \mathrm{d}\left(e^{\Delta} Z\right) &= e^{\Delta} Y' \wedge F \end{aligned} \quad \longleftrightarrow \quad L_K K = 0$$

Generalised Calibrations
M-theory

- Exceptional Sasaki-Einstein Structure: Vector structure

$$K = \xi + \left(\iota_\xi A - e^{\Delta} Y'\right) + \left(e^{\Delta} Z - e^{\Delta} A \wedge Y' + \tfrac{1}{2}\iota_\xi A \wedge A\right)$$

- World-volume pullback of the bilinears

$$\mathcal{K}_m = -i\left(\bar{\chi}_1 + \chi_2^T\right)\gamma_m\left(\chi_1 - \chi_2^*\right) = \xi_m$$

$$\omega_{m_1 m_2} = e^{\Delta}\,\bar{\chi}\gamma_7\gamma_{m_1 m_2}\chi = e^{\Delta} Y'$$

$$\Sigma_{m_1\ldots m_5} = e^{\Delta}\,\bar{\chi}\gamma_7\gamma_{m_1\ldots m_5}\chi = e^{\Delta} Z$$

- Calibration form

$$\Phi_{\mathrm{M5}} = e^{\Delta} Z - e^{\Delta} A \wedge Y' + \tfrac{1}{2}\iota_\xi A \wedge A$$

- Where the closure condition comes from? Integrability conditions

Generalised Calibrations

M-theory

- Calibration form

$$\Phi_{\mathrm{M5}} = e^{\Delta} Z - e^{\Delta} A \wedge Y' + \tfrac{1}{2}\iota_\xi A \wedge A$$

- Closure

$$\begin{aligned} \mathrm{d}\Phi_{\mathrm{M5}} &= \mathrm{d}(e^{\Delta} Z) - \mathrm{d}(e^{\Delta} A \wedge Y') + \tfrac{1}{2}\mathrm{d}(\iota_\xi A \wedge A) = \\ &= e^{\Delta} Y' \wedge F - F \wedge e^{\Delta} Y' + \iota_\xi F \wedge A + \tfrac{1}{2}\mathrm{d}(\iota_\xi A) \wedge A + \tfrac{1}{2}\iota_\xi A \wedge F = \\ &= \iota_\xi F \wedge A + \tfrac{1}{2}\mathcal{L}_\xi A \wedge A - \tfrac{1}{2}\iota_\xi F \wedge A - \tfrac{1}{2} A \wedge \iota_\xi F = 0 \end{aligned}$$

- We found the calibration form from the Exceptional structure
- Using integrability of the structures we can write closure of the forms
- It is possible to relate different probe configurations to the same structure

Summary and Conclusions

- Exceptional Sasaki-Einstein structures encode informations about brane calibrations on AdS backgrounds
- One can construct all the calibration forms from them
- Integrability conditions are equivalent to the closure of the forms
- The calibrations for all brane configurations are included in HV structures
- For branes wrapping purely internal cycles, calibrations correspond to the vector structure


Thank You