Aspects of η and λ models

Recent Advances in T/U-dualities and Generalized Geometries, Rudjer Boskovic Institute

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Appadu, Hollowood, Price, DT [1706.0xxxx]

Demulder, Dorigoni, DT [1604.07851]

Sfetsos, Siampos, DT [1506.05784]

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4. Generalised λ models, symmetries, S-matrix and quantisation

Motivation

Shed light on dualities

- An applications of generalised T-dualities
- A regularisation of non-abelian duality

Connection to DFT

- Relation between PL and non-Abelian and SS reduced DFT
- Beyond sugra: η deformation modified supergravity

Harness the power of integrability

- 2d QFT's as toys to probe non-perturbative physics
- Close interplay between T-duality and integrability

Contents

1. New integrable 2d QFTs η and λ models

Recap: the Principal Chiral Model

$$S = \frac{1}{2\pi}\int_{\Sigma} d^2\sigma \, Tr\left(g^{-1}\partial_+ g g^{-1}\partial_- g\right) \ , \quad g : \Sigma \to SU(N)$$

- e.g. $SU(2)$ is σ-model into S^3 with $SU(2)_L \times SU(2)_R$ symmetry
- **Integrable**: Lax formulation and ∞ conserved charges

$$\mathcal{L}(z) = \frac{1}{1-z^2}J + \frac{z}{1-z^2}\star J \ , \quad d\mathcal{L} - \mathcal{L}\wedge\mathcal{L} = 0 \ , \quad T(z) = Pexp\int d\sigma \mathcal{L}_\sigma$$

- Non-conformal proto-QCD model (but no instantons), factorised S-matrix [Zamolodchikov,Zamolodchikov;Polyakov,Wiegmann] and exact mass gap [Neidermayer , P Hasenfratz (1946-2016)]

$$\mathbb{S}[\theta] = S_{SU(2)} \otimes S_{SU(2)} \qquad \frac{m}{\Lambda_{\bar{MS}}} = \frac{N}{\sqrt{\pi e}} 2^{\frac{3}{2}} \sin\frac{\pi}{N}$$

XXZ - sigma model

- Deform to a σ-model on a squashed S^3 [Cherednik '81]:

$$S = \frac{1}{2\pi}\int_\Sigma d^2\sigma \, Tr\left(g^{-1}\partial_+ g g^{-1}\partial_- g\right) + C J_+^3 J_-^3$$

- Integrable but $SU(2)_L \times SU(2)_R \to SU(2)_L \times U(1)_R$
- Non-local charges recover semi-classical version of (affine extension of) $\mathcal{U}_q(\mathfrak{sl}_2)$ [Kawaguchi, Matsumoto, Yoshida '11, '12]

$$\{Q_R^+, Q_R^-\}_{P.B.} = \frac{q^{Q_R^3} - q^{-Q_R^3}}{q - q^{-1}}, \quad q = \exp\left(\frac{\sqrt{C}}{1+C}\right)$$

- Important subtleties:
  1. Lax involves **trig** functions + affine $\mathcal{U}_q(\mathfrak{su}_2)$ **principal** gradation
  2. $-1 < C < 0$ is a UV safe; $C > 1$ large couplings in both IR and UV

Yang-Baxter and η Deformations

Integrable models [Klimcik '02] based on modified Yang-Baxter eq

$\mathcal{R}$-matrix: Solution of classical (modified) YB equation:

$$[\mathcal{R}A, \mathcal{R}B] - \mathcal{R}([\mathcal{R}A, B] + [A, \mathcal{R}B]) = -c^2[A, B] \ , \quad \forall A, B \in \mathfrak{g}$$

$\mathcal{R}$-matrix defines an integrable deformed σ-model

$$S_\eta = \frac{1}{2\pi t}\int_\Sigma d^2\sigma Tr\left(g^{-1}\partial_+ g, \frac{1}{1-\eta\mathcal{R}} g^{-1}\partial_- g\right)$$

For $SU(2)$ this is the squashed three sphere + pure gauge B-field with $C \sim \eta^2$

- Important subtleties:
  1. Lax involves **rational** functions + affine $\mathcal{U}_q(\mathfrak{sl}_2)$ **homogenous** gradation
  2. UV safe has η pure imaginary; but ok for $SU(2)$

Yang-Baxter and η Deformations II

- Cosets and super-cosets e.g. $AdS_5 \times S^5$ superstring [Delduc, Magro, Vicedo 1309]
- κ-symmetric, solves **modified** SUGRA [Orlando et al 1607, Arutyunov et al. 1511]
- Weyl invariant (solve SUGRA) if unimodular [Borsato and Wulff 1608]

$$\mathcal{R}^B{}_A F^A{}_{BC} = 0$$

- Relation to DFT currently being investigated [Sakamoto et al; Baguet et al]
- Case 1: $c^2 = -1$ η Deformations
- Case 2: $c = 0$ Yang-Baxter Deformations

λ-deformations: The Sfetsos Procedure

Rather similar to the Buscher procedure this recipe produces integrable λ deformations [Sfetsos 1312] as a regularisation of non-Abelian T-duality

1. **Double** the d.o.f.: $\kappa^2 S_{PCM}[\tilde{g}] + k S_{WZW}[g]$
2. **Gauge** G_L in PCM and G_{diag} in WZW
3. **Gauge Fix** $\tilde{g} = 1$
4. **Integrate out** non-propagating gauge fields

$$S_\lambda = k S_{WZW} + \frac{k}{2\pi} \int Tr(g^{-1}\partial_+ g \frac{1}{\lambda^{-1} + Ad_g} \partial_- g g^{-1})$$

$$\lambda = \frac{k}{\kappa^2 + k}$$

λ Deformations interpolate between CFT and non-Abelian T-duals

Nice behaviour in limits of small and large deformations:

- $\lambda \to 0$: current bilinear perturbation

$$S_\lambda|_{\lambda\to 0} \approx kS_{WZW} + \frac{k}{\pi}\int \lambda J_+^a J_-^a + \mathcal{O}(\lambda^2)$$

- $\lambda \to 1$: non-Abelian T-dual of PCM

$$S_\lambda|_{\lambda\to 1} \approx \frac{1}{\pi}\int \partial_+ X^a (\delta_{ab} + f_{ab}{}^c X_c)^{-1} \partial_- X^b + \mathcal{O}(k^{-1})$$

The T-dual is recovered because in this limit the gauged WZW in the Sfetsos Procedure becomes a Lagrange multiplier term of the Buscher Procedure

λ Commentary

- λ deformations **do** solve SUGRA with appropriate RR fields [Sfetsos DT]
- Quantum group symmetry expected with $q = e^{\frac{i\pi}{k}}$ [Hollowood et al]
- Also applied to cosets [Sfetsos], supercosets [Hollowood et al]
- One-loop marginal deformation in case of $PSU(2,2|4)$! [Appadu, Hollowoodl]

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2. Relation with generalised T-duality

η, λ and Poisson-Lie

η and λ connected by generalised Poisson Lie T-duality

[Vicedo 1504; Hoare & Tseytlin 1504; Siampos Sfetsos DT 1506; Klimcik 1508]

- Modified conservation law for currents of broken G_R in η-model:

$$d \star J_a = \tilde{f}^{bc}{}_a \, J_b \wedge J_c$$

- $\tilde{f}^{bc}{}_a$ structure constants for $\mathfrak{g}_{\mathcal{R}}$

$$[A, B]_{\mathcal{R}} = [\mathcal{R}A, B] + [A, \mathcal{R}B]$$

- Mathematically $\mathfrak{g} \oplus \mathfrak{g}_{\mathcal{R}} \simeq \mathfrak{g}^{\mathbb{C}}$ defines a Drinfel'd Double
- $\star\mathcal{J}$ pure gauge in a dual algebra (Field Equations $\Leftrightarrow$ Bianchi identity)
- So although not isometric just the right structure for PL T-duality [Klimcik Severa]

PLT Pairs of Sigma models

- PL T-duality equivalence between two σ-models

$$S[g] = \frac{1}{2\pi t}\int d^2\sigma L_+^T (E - \Pi)^{-1} L_- \,, \quad g \in \mathcal{G}\,,$$
$$\tilde{S}[\tilde{g}] = \frac{1}{2\pi t}\int d^2\sigma \tilde{L}_+^T (E^{-1} - \tilde{\Pi})^{-1} \tilde{L}_- \,, \quad \tilde{g} \in \tilde{\mathcal{G}}\,.$$

$$a_a{}^b = \langle g^{-1} T_a g, \tilde{T}^b\rangle \,, \quad b^{ab} = \langle g^{-1}\tilde{T}^a g, \tilde{T}^b\rangle \,, \quad \Pi = b^T a$$

For η deformation $E = \eta^{-1} - \mathcal{R}$.

A doubled formalism for PL T-duality

- PL Dual Pairs follow from a first order "doubled formalism" chiral-WZW

  [Klimcik & Severa, Sfetsos, Hull & Reid-Edwards]

$$S = \int_{\Sigma} d^2\sigma - \mathcal{H}_{AB}\mathbb{L}^A_\sigma \mathbb{L}^B_\sigma + \eta_{AB}\mathbb{L}^A_\sigma \mathbb{L}^B_\tau + \int_{\mathcal{M}_3} f_{AB}{}^D \eta_{DC}\mathbb{L}^A \wedge \mathbb{L}^B \wedge \mathbb{L}^C$$

The Doubled σ-model lives on the Drinfeld Double group [Klimcik & Severa]

- The Drinfeld Double is a Lie Algebra $\mathcal{D}$ which can be decomposed as the sum $\mathcal{D} = \mathcal{G} + \tilde{\mathcal{G}}$ for two maximally isotropic sub-algebras. If $T_A = \{T_a, \tilde{T}^a\}$ then

$$\langle T_a | T_b \rangle = \langle \tilde{T}^a | \tilde{T}^b \rangle = 0 \Rightarrow \langle T_A | T_B \rangle = \eta_{AB}$$

- Covariant version [Driezen,Sevrin,Thompson]

PLT and DFT?

- The generalised metric is dressed and so has coordinated dependence

$$\mathcal{H}(\mathbb{X})_{MN} = \mathcal{H}_{AB}\mathbb{L}_M^A\mathbb{L}_N^B$$

-- $\mathcal{D} = u(1)^d + u(1)^d \Rightarrow$ **Abelian T-duality** $\Rightarrow \mathcal{H}_{MN}$ constant
-- $\mathcal{D} = \mathcal{G} + u(1)^d \Rightarrow$ **non-Abelian T-duality** $\Rightarrow \mathcal{H}_{MN}(x)$ on-section
-- $\mathcal{D} = \mathcal{G} + \tilde{\mathcal{G}} \Rightarrow$ **Poisson-Lie T-duality** $\Rightarrow \mathcal{H}_{MN}(x, \tilde{x})$ beyond-section

β-function of $\mathcal{H}_{AB}$ implies scalar potential of gauged supergravity **[Avramis, Derendinger, Prezas; Sfetsos-Siampos-DT]**

- See Scherk Schwarz reduced DFT

η, λ and Poisson-Lie

- So back to the η model we can PL dualise
- Not quite enough...
- Analytic continue certain Euler angles and deformation parameters

$$\eta \to i\frac{1-\lambda}{1+\lambda} \,, \quad t \to \frac{\pi(1+\lambda)}{k(1-\lambda)}$$

- Acting on the parameter q we have

$$q = e^{\eta t} \leftrightarrow q = e^{\frac{i\pi}{k}}$$

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3. Quantum aspects of η model

Perturbative headaches

- Perturbation theory is **great!**

$$\left.\frac{(g-2)}{2}\right|_{theory} = 0.00115965218178(77) \quad \text{[Aoyama et al.]}$$

$$\left.\frac{(g-2)}{2}\right|_{experiment} = 0.00115965218073(28) \quad \text{[Gabrielse et al.]}$$

- Perturbation theory is an uncontrolled disaster!

$$E^{pert} = \sum_{n=0}^{\infty} c_n (g^2)^n \,, \qquad c_n \sim n!$$

generic factorial growth due to # of diagrams (and in QFT loop momenta)

Can we make formal sense of perturbation theory?
Do the divergences hide some real physics?

Borel Resumation

Borel-Laplace resumation gives meaning to divergent sums

$$\mathcal{B}[E^{pert}](t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n \,,$$

$$\mathcal{S}[E^{pert}](g^2) = \frac{1}{g^2} \int_0^{\infty} \mathcal{B}[E^{pert}](t) e^{-\frac{t}{g^2}} \mathrm{d}t \,.$$

Extend over $\mathbb{C}$ to avoid poles $\Rightarrow$ directional resumation

$$\mathcal{S}_\theta[E^{pert}](g^2) = \frac{1}{g^2} \int_0^{e^{i\theta}\infty} \mathcal{B}[E^{pert}](t) e^{-\frac{t}{g^2}} \mathrm{d}t \,,$$

Finite but ambiguous result

$$\Delta E = \mathcal{S}_{\theta>0} - \mathcal{S}_{\theta<0} \sim i e^{-\frac{S}{g^2}}$$

The resurgence idea

Non-perturbative sector cancels perturbative ambiguities

- **Bogomolny; Zinn-Justin**: instanton–anti-instanton $[I\bar{I}]$ ambiguity from quasi-zero model integration exactly cancels perturbative ambiguity!
- Just start of a whole structure of ambiguity cancellations e.g. late terms in $[I]$ cancel early terms in $[II\bar{I}]$
- Semi-classical approximation $\rightarrow$ Écalle trans-series

$$E(g^2) = E^{pert} + \sum_i e^{-\frac{S_i}{g^2}} \sum_{k>0} c_k^{(i)} (g^2)^k \,,$$

- Location of poles in Borel plane precisely tied to semi-classical saddles
- Not new **[Bender and Wu; Reeve Stone; Berry Howls; Voros]** but lots of activity in QM; (topological) String Theory, Matrix models, QFT, Localisation **[Zinn-Justin Jentschura; Dunne, Unsal; Pasquetti Schiappa; Ancieto, Schiappa, Vonk; Mariño; Cherman, Dorigoni, Unsal; Gaiotto Moore Neitzke]**

The resurgence puzzles

Intriguing idea but several puzzles remain

- QFT vs. QM
- Multiple couplings
- Theories without instantons
- Role of complex saddles

η-deformed $SU(2)$ theory gives a simple tractable example to study these ideas

Strategy

1. Reduce in a clever way to a QM
2. Study the large order behaviour of the QM
3. Identify non-perturbative objects that give rise to Borel poles
4. Exhibit the 2d QFT origin of these NP saddles

The perturbative sector 1

- Put $SU(2)$ η-model on $\mathbb{R} \times S^1$ with twisted b.c.

$$g(t, x + L) = e^{iH} g(t, x) e^{-iH} , \quad H = \frac{\pi}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  -- Idea [Cherman et al 1403] PCM physics smoothly connected physics between large and small radius (provable for large N $O(N)$ [Sulejmanpasic 1610])
  -- Here an assumption; post hoc we find a consistent story

- After truncating to lightest modes we get the Whittaker-Hill QM

$$-g^4 \frac{d^2\Psi}{dx^2} + (V(x) - g^2 E)\Psi = 0 , \quad V(x) = \sin^2(x) + \eta^2 \sin^4(x)$$

The perturbative sector 2

- We use a WKB method to find the perturbative ground state energy as a solution to Riccati equation

$$\Psi = \exp(-\frac{S(x)}{g^2}) \Rightarrow g^2 S''(x) + S'(x)^2 - (V - g^2 E) = 0$$

- Do a double expansion and solve order by order in g^2:

$$S(x) = \sum_n g^{2n} S_n(x) \ , \quad E = \sum_n g^{2n} E_n$$

- Easily done on machine to $(g^2)^{100}$. e.g. for $\eta = \frac{1}{2}$

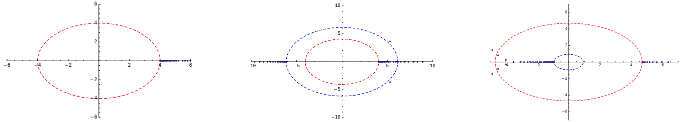
$$E = 1 - \frac{1}{16} g^2 - \frac{61}{256} g^4 + \frac{77}{4096} g^6 + \cdots +$$

The perturbative sector 3

- An all orders approximant: Borel Padé

$$B_N(t) = \sum_{n=1}^{N} \frac{1}{n!} E_{n-1} t^n \approx \frac{P_n(t)}{1 + Q_m(t)}\ , \qquad n + m = N$$

- Identify branch cuts in full Borel transform as accumulated poles of the Borel Padé



$\eta = 0$ $\qquad$ $\eta = 1/5$ $\qquad$ $\eta = 3/4$

Figure: Singularities of the diagonal Borel-Padé approximant $B_{75,75}(t)$ in the complex Borel plane for different values of $\eta = 0,\ \eta = \frac{1}{5},\ \eta = \frac{3}{4}$. The dashed red circle denotes $|t| = |S_{\bar{I}I}|$, while the dashed blue circle denotes $|t| = |S_{\mathbb{C}I}|$.

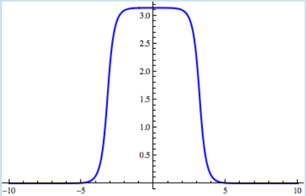
The non-perturbative sector 1

Location of Borel Poles corresponds to [I $\bar{I}$] and complex instanton events

Instanton

$$\theta_I(t) = \pi - \text{arccos}\left[\frac{\sqrt{1+\eta^2}\,\tanh(t-t_0)}{\sqrt{1+\eta^2\tanh^2(t-t_0)}}\right]$$

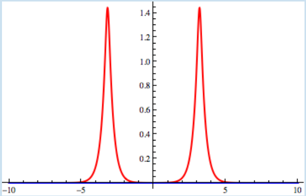
$$S_I = \left(1 + (\eta+\eta^{-1})\,\text{arctan}(\eta)\right)$$



Complex Instanton

$$\theta_{CI}(t) = -\frac{\pi}{2} + i\text{arctanh}\left[\sqrt{1+\eta^2}\,\cosh(t-t_0)\right]$$

$$S_{CI} = \left(1 - (\eta+\eta^{-1})\text{arccot}(\eta)\right)$$



QFT origin of Non-Perturbative Contributions

- Positive Borel poles match non-perturbative objects: Unitons Uhlenbeck
- Not instantons ($\pi_2[SU(N)] = 0$), non BPS, finite action solutions
- Depend on a single holomorphic function $f(z)$

$$U_\eta = \frac{-i}{\sqrt{(1+|f|^2)^2 + \eta^2(1-|f|^2)^2}} \begin{pmatrix} \sqrt{1+\eta^2}(1-|f|^2) & 2\bar{f} \\ e2f & -\sqrt{1+\eta^2}(1-|f|^2) \end{pmatrix} .$$

$$g^2 S_\eta[U_\eta] = 2S_I .$$

- Lump of Lagrangian density fractionates on $\mathbb{R} \times S^1$ with twisted b.c.

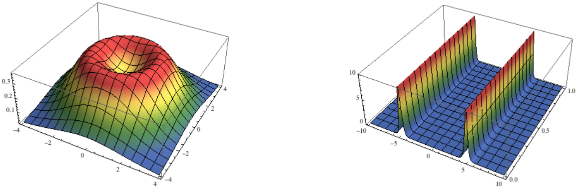


Figure: Lagrangian density of $SU(2)$ η-deformed uniton on $\mathbb{R}^2$ (left) and $\mathbb{R} \times S^1$ (right)

The non-perturbative sector 3

- Negative Borel poles surprising: *complex* Unitons

$$U^c_\eta = \frac{1}{\sqrt{4|f|^2 - (1+\eta^2)(1+|f|^2)^2}} \begin{pmatrix} \sqrt{1+\eta^2}(1+|f|^2) & 2i\bar{f} \\ 2if & -\sqrt{1+\eta^2}(1+|f|^2) \end{pmatrix} .$$

$$g^2 S_\eta[U^c_\eta] = S_{Cl} \,, \quad S_{Cl} = (1 - (\eta + \eta^{-1})\text{arccot}(\eta))$$

- Again fractionates on $\mathbb{R} \times S^1$ with twisted b.c.
- For $\eta \to 0$ the action diverges and these drop out of the physics whilst at critical value $\eta_c \sim 0.27$ these dominate over the real saddles

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4. Generalised λ models, symmetries, S-matrix and quantisation

Generalised λ & YB-λ Theories

- Sfetsos Procedure can be generalised by replacing PCM:

$$kS_{WZW}[g] + S[\tilde{g}] = \int Tr\left(\tilde{g}^{-1}\partial_+ g \Theta \tilde{g}^{-1}\partial_- g\right)$$

- λ now a matrix Λ:

$$S_\lambda = kS_{WZW} + \frac{k}{2\pi}\int Tr(g^{-1}\partial_+ g \frac{1}{\Lambda^{-1} + Ad_g}\partial_- g g^{-1})$$

$$\Lambda = 1 + k^{-1}\Theta$$

- **Idea**: if Θ defined integrable PCM, Λ can define an integrable theory

Generalised λ & YB-λ Theories for $SU(2)$

λ-XXZ Model

$$\Theta = \mathrm{diag}(\xi^{-1}, \xi^{-1}, \lambda^{-1})$$

Trigonometric Lax

$$\mathcal{L}_\sigma = f_+[z]^a \mathcal{J}_+^a T^a - f_-[z]^a \mathcal{J}_-^a T^a$$

RG invariant

$$\gamma'^2 = \frac{k^2}{4} \frac{(1-\xi^2)(1-\lambda)^2}{\lambda^2 - \xi^2}$$

λ-YB Model

$$\Theta = I + \frac{1}{kt}(1 - \eta \mathcal{R})^{-1}$$

Rational Lax

$$\mathcal{L}_\sigma = (c_+ + d\mathcal{R})\mathcal{J}_+ + (c_- + d\mathcal{R})\mathcal{J}_-$$

RG invariant

$$\Sigma = \frac{2\pi\eta\lambda}{k(1-\lambda)}$$

"Non ultra-local" i.e. central term in current algebra

$$\{\mathcal{J}_\pm^a(x), \mathcal{J}_\pm^b(y)\} = f_{ab}{}^c \mathcal{J}_\pm^c(x)\delta_{xy} \pm \frac{k}{2\pi}\delta^{ab}\delta'_{xy}$$

Classical Symmetries

- Expand monodromy to find symmetries but need to determine expansion points!

$$T(z) = P\exp\left(-\int \mathcal{L}_\sigma(z)\right)$$

- Determine Maillet r/s algebra

$$\{\mathcal{L}_\sigma^{\underline{1}}, \mathcal{L}_\sigma^{\underline{2}}\} = [r(z_1, z_2), \mathcal{L}_\sigma^{\underline{1}} + \mathcal{L}_\sigma^{\underline{2}}]\delta_{12} + [s(z_1, z_2), \mathcal{L}_\sigma^{\underline{1}} - \mathcal{L}_\sigma^{\underline{2}}]\delta_{12} - 2s(z_1, z_2)\delta'_{12}$$

- Locate special points $z_\star$ where $\lim_{\epsilon\to 0} r(z_\star, z_\star + \epsilon) = \textit{finite}$

Charges and Symmetries

- Special points associated to Quantum Group Symmetries
- e.g. For $\lambda - YB$ model at $c(z_\star) = i\, d(z_\star)$ we find

$$Q^3 \sim \int \mathcal{J}_0^3 \,, \quad Q^\pm \sim \int (\mathcal{J}_0^1 \pm i\mathcal{J}_0^2) \exp\left[-i\Sigma \int_{-\infty}^{\pm x} \mathcal{J}_0^3(\pm y)dy\right]$$

$$q = \exp\left(\frac{2\pi\eta\lambda}{k(1-\lambda)}\right) = e^\Sigma \quad \textbf{Homogenous Gradation}$$

- For $\lambda - XXZ$ model similar with $q = \exp[\pi\sqrt{\gamma'^2}]$ **Principal Gradation**
- QG parameters are RG invariant
- Second quantum group point given by KM currents with

$$q'_{cl} = \exp\left(\frac{i\pi}{k}\right)$$

Exact S-Matrix

Based on symmetries, limits and RG behaviour, we find conjectured form for S-matrices using known blocks

- λ-XXZ Model in UV Safe Domain $\gamma'^2 < 0$ Bernard LeClair

$$\mathcal{S}_{\lambda-XXZ} = S_{SG}(\theta, \gamma') \otimes S^{(k)}_{RSOS}(\theta)$$

- λ-XXZ Model Other Domain (periodic in rapidity)

$$\mathcal{S}_{\lambda-XXZ} = S_{p}(\theta, \Sigma) \otimes S^{(k)}_{RSOS}(\theta)$$

- λ-YB Model (periodic in rapidity, parity broken)

$$\mathcal{S}_{\lambda-XXZ} = S_{h}(\theta, \Sigma) \otimes S^{(k)}_{RSOS}(\theta)$$

'Proving' S-matrix I

- Non-ultra-local *i.e.* δ' makes conventional techniques (QISM) inapplicable
- Alleviation Faddeev-Reshetikhin takes a limit, modifies UV but same IR properties

$$k \to 0 \, , \quad \frac{k}{\xi} \, , \frac{k}{\lambda} \; \textit{fixed}$$

- In this limit the Lax connection becomes ultra-local ($s(z,w) \to 0$) and can be regularised, and quantised, on a lattice
- Obtain a lattice theory, XXZ anisotropic spin chain.

$$H_{\frac{1}{2}} = \sum_{n=1}^{N} \left(\sigma_n^1 \sigma_{n+1}^1 + \sigma_n^2 \sigma_{n+1}^2 + \cos \gamma \sigma_n^3 \sigma_{n+1}^3 \right)$$

- Actually need a spin $S = \frac{k}{2}$ chain and identify

$$\gamma = \frac{\pi}{\gamma'} - k$$

'Proving' S-matrix II

- Ground state using TBA Kirillov-Reshetikhin find Dirac Sea dominated by k-Bethe strings whose density $\rho(z)$ obeys integral equation

$$\rho(z) + \rho_h(z) + \frac{1}{\pi}\int K(z-y)\rho(y)dy = \epsilon(z)$$

- Holes with density ρ_h are excitations above the ground state
- **Amazing fact**, these excitations scatter relativistically with a kernel

$$\tilde{K}(z) = \frac{d}{dz}LogS(z) = \int_0^\infty \cos(z\omega)\left(\coth(k\omega) + \coth(\gamma'\omega)\right)\tanh \pi\omega$$

- This corresponds exactly to the S-matrix of the λ-XXZ Model

Conclusions

Conclusions

- η and λ models are interesting 2d QFTs
- Provide a realisation of Poisson Lie duality
- New perspective on non-Abelian T-duality
- η model provided an interesting playground for resurgence
- λ model has tractable multi-parameter generalisations

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Appendix: S-matrix Technology

Rapidity

$$E = m \cosh \theta \ , \quad P = m \sinh \theta$$

Axioms:

1. *Factorization* 2-body factorisation, no particle production
2. *Analyticity*. Only poles along the imaginary axis $0 < Im\theta < \pi$ associated to stable bound states.
3. *Hermitian analyticity*
$$S_{ij}^{kl}(\theta^*)^* = S_{kl}^{ij}(-\theta) \ .$$
4. *Unitarity*
$$\sum_{kl} S_{ij}^{kl}(\theta) S_{mn}^{kl}(\theta)^* = \delta_{im}\delta_{jn} \ , \qquad \theta \in \mathbb{R} \ .$$
5. *Crossing*
$$S_{ij}^{kl}(\theta) = \mathcal{C}_{kk'} S_{k'\,i}^{lj'}(i\pi - \theta) \mathcal{C}_{j'\,j}^{-1} = S_{\bar{k}i}^{l\bar{j}}(i\pi - \theta) \ ,$$
where $\mathcal{C}$ is the charge conjugation matrix.

Appendix: Gradation I

$$[H_i, E_j] = a_{ij}E_j, \quad [H_i, F_j] = -a_{ij}F_j\ , \quad [E_i, F_j] = \delta_{ij}H_j$$

Generalised Cartan matrix a_{ij} has off diagonal elements equal -2.
$K = H_0 + H_1$ is central. $K = 0$, i.e. centreless representations $\widehat{\mathfrak{su}(2)}$ becomes the loop algebra. Reps are the tensor of an $\mathfrak{su}(2)$ rep and functions of a variable z. Gradation is the relative action in $\mathfrak{su}(2)$ space and z-space.

homogenous gradation

$$E_1 = T^+, \quad F_1 = T^-, \quad E_0 = z^2T^-, \quad F_0 = z^{-2}T^+, \quad H_1 = -H_0 = T^3$$

. *principal* gradation

$$E_1 = zT^+, \quad F_1 = z^{-1}T^-, \quad E_0 = zT^-, \quad F_0 = z^{-1}T^+, \quad H_1 = -H_0 = T^3$$

Appendix: Homogenous Gradation

	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\uparrow$	$+2$	$\mathfrak{Q}_2^+$	$\mathfrak{Q}_2^3$	$\mathfrak{Q}_2^-$
$z_* = +i\eta$	$+1$	$\tilde{\mathfrak{Q}}^+$	$\mathfrak{Q}_1^3$	$\mathfrak{Q}_1^-$
	0	$\mathfrak{Q}^+$	$\mathfrak{Q}^3 = -\tilde{\mathfrak{Q}}^3$	$\mathfrak{Q}^-$
$z_* = -i\eta$	-1	$\mathfrak{Q}_{-1}^+$	$\mathfrak{Q}_{-1}^3$	$\tilde{\mathfrak{Q}}^-$
$\downarrow$	-2	$\mathfrak{Q}_{-2}^+$	$\mathfrak{Q}_{-2}^3$	$\mathfrak{Q}_{-2}^-$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure: The charges and their grades for the expansion of the monodromy around the pair of special points $z = \pm i\eta$. The blue/red and positive/negative graded charges are associated to $\pm i\eta$, respectively. The red and blue charges generate the affine quantum group in homogenous gradation and all the other charges are obtained by repeated Poisson brackets of these charges.

Appendix: Principal Gradation

$\widehat{\mathfrak{su}(2)}_p$.

	$\vdots$	$\vdots$		$\vdots$
$\uparrow$	$+2$		$\mathfrak{Q}^3_2$	
$z_* = \infty$	$+1$	$\mathfrak{Q}^+$		$\tilde{\mathfrak{Q}}^-$
	0		$\mathfrak{Q}^3 = -\tilde{\mathfrak{Q}}^3$	
$z_* = -\infty$	-1	$\tilde{\mathfrak{Q}}^+$		$\mathfrak{Q}^-$
$\downarrow$	-2		$\mathfrak{Q}^3_{-2}$	
	$\vdots$	$\vdots$		$\vdots$

Figure: The charges and their grades for the expansion of the monodromy around the pair of special points $z = \pm\infty$ (or $0, \infty$ with a multiplicative spectral parameter). The blue/red and positive/negative graded charges are associated to $\pm\infty$, respectively. The red and blue charges generate the affine quantum group in principal gradation and all the other charges are obtained by repeated Poisson brackets of these charges.

RG in YB-λ model

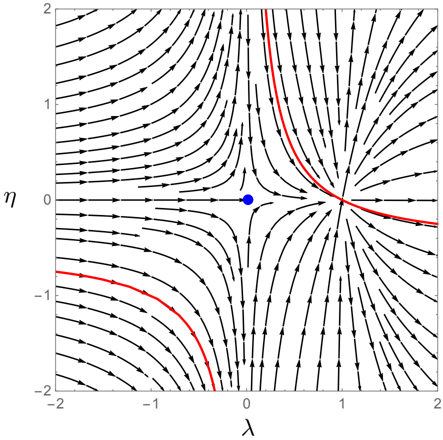

Figure: The RG flow of the YB lambda model (flows towards the IR). The WZW fixed point is the blue dot in the middle. The red curved is an example of a cyclic trajectory which has a jump from $\eta = +\infty$ to $-\infty$ at $\lambda = 0$ and a jump from $\lambda = -\infty$ to $\lambda = +\infty$.

RG in η-λ model

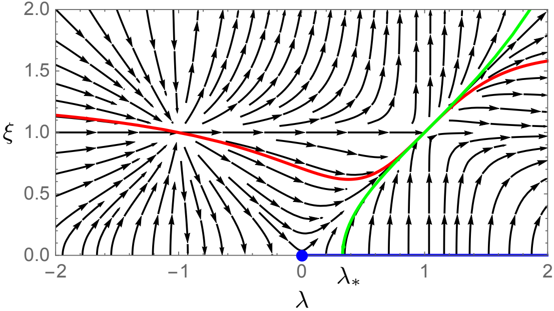

Figure: The RG flow (to the IR) of the XXZ lambda model. The WZW fixed point is identified by the blue blob. The blue line is a line of UV fixed points. The green curve is a UV safe trajectory that has $\gamma' \in \mathbb{R}$. The red curve is a cyclic RG trajectory with $\gamma' = i\sigma$, $\sigma \in \mathbb{R}$. The trajectory has a jump in the coupling λ from $-\infty$ to ∞, but is continuous in $1/\lambda$.