Poisson-Lie T-duality and generalized isometries

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- perhaps the right "general T-duality"

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Courant algebroid: vector bundle $E \to M$, symmetric pairing $\langle\, , \rangle$ anchor map $\rho : E \to TM$, bracket $[\, ,] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ such that ($\forall s, t, u \in \Gamma(E)$ and $f \in C^\infty(M)$)

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Examples

- exact CAs

$$0 \to T^*M \to E \to TM \to 0$$

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- Lie algebras with invariant symmetric pairing ($M =$ point)

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Generalized metric: a vector subbundle $V^+ \subset E$ s.t. $\langle \, , \, \rangle$ is positive-def. on V^+ and negative-def. on $V^- := (V^+)^\perp$
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Example

A generalized metric in an exact CA $E \to M$
$=$ a Riemannian metric g and a closed 3-form H on M
(i.e. the data needed for a 2-dim σ-model)

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PL T-duality

If (M_1, g_1, H_1) and (M_2, g_2, H_2) are obtained by pulling back the same gen. metric $\tilde{V}^+ \subset \tilde{E}$ then the corresponding 2-dim σ-models are (almost) isomorphic as Hamiltonian systems

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(A better description: A multiplicative gerbe over D trivial on G_i's, acting on a gerbe on P)

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- A generalized metric $V^+ \Rightarrow$ a Hamiltonian $\mathcal{H}$ on $L_{CA}E$

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$L_{CA}\tilde{E}$ is the reduction of a finite-codimension coisotropic submanifold in $L_{CA}E_i$, i.e. $L_{CA}E_{1,2}$ are **almost isomorphic as Hamiltonian systems**

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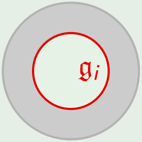
$L_{\mathrm{CA}}(\mathfrak{d} \times D/G_i)$: 

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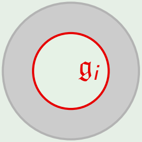

The Hamiltonian is $\mathcal{H}(A) = \frac{1}{2}\int_{S^1}\langle A_\sigma, \mathbf{V}A_\sigma\rangle\, d\sigma$ where $\mathbf{V}$ is the reflection w.r.t. $V^+ \subset \mathfrak{d}$.

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$$S(A) = \int_Y \Big(\frac{1}{2}\big\langle A, dA\big\rangle + \frac{1}{6}\big\langle [A,A], A\big\rangle\Big) \qquad A \in \Omega^1(Y, \mathfrak{d})$$

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σ-model type boundary condition

needs a pseudo-Riemannian metric on $\Sigma \subset \partial Y$ and $V^+ \subset \mathfrak{d}$

$$*(A|_\Sigma) = \mathbf{V} A|_\Sigma$$

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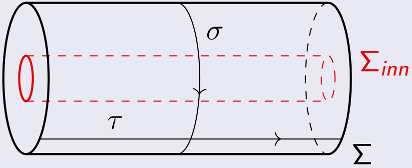
Hollow cylinder: The σ-model with the target D/G



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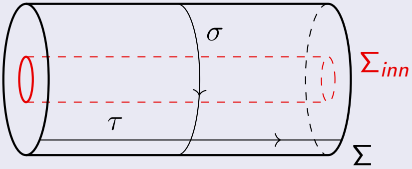

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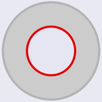

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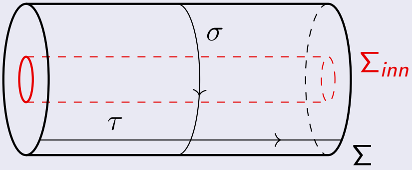

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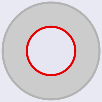

Full cylinder: The duality-invariant part (reduced phase space)

Ricci flow
[P.Š., Fridrich Valach, 2016]

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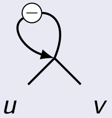
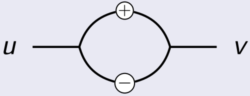
Generalized Ricci flow (of a generalized metric)

$$\frac{dV^+}{dt} = T_{V^+} : V_+ \to V_- \qquad \langle T_{V^+} u, v \rangle = -2\,\mathrm{GRic}_{V^+}(u,v)$$

$\mathrm{GRic}_{V^+}(u,v) :=$ 

$-$ 

Compatible with pull-backs $\Rightarrow$ PL T-duality is compatible with Ricci flow

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Equivalently: a Lagrangian dg submanifold of the dg symplectic manifold $\mathcal{E}$

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Example

When $E = (T \oplus T^*)M \to M$ is exact given by $H \in \Omega^3(M)_{\text{closed}}$ and $\omega \in \Omega^2(N)$ ($N \subset M$) s.t. $d\omega = H|_N$ then

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Generalized isometries

A **generalized isometry** between $V_1^+ \subset E_1 \to M_1$ and $V_2^+ \subset E_2 \to M_2$ is a Dirac relation C s.t. $(\mathbf{V}_1 \times \mathbf{V}_2)C = C$

(Almost) isomorphisms of Hamiltonian systems

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- How to generate examples besides PL T-duality?
- Composition and global issues (derived geometry?)

Open problems: higher dimensions

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Problem for $n \geq 3$

Make it compatible with gauge symmetries, find non-trivial dualities of (higher) gauge theories

Open problems: quantization

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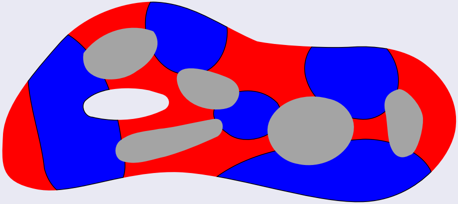

3-dim Y
Σ = gray part of ∂Y
K finite Abelian group
$f : H^1(\Sigma, \partial\Sigma_{red}; K) \to \mathbb{C}$
(Boltzmann weight)

$$Z_{red}(f, K) := \sum_{\alpha \in H^1(Y, \partial Y_{red}; K)} f(i^*\alpha)$$

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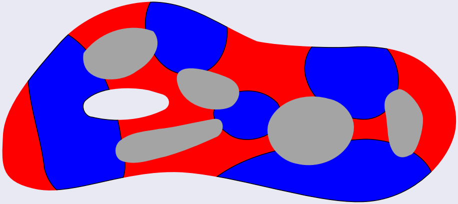

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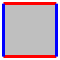
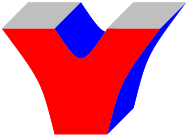
$H = Z($  $)$

Hopf algebra

$\mathfrak{g}, \mathfrak{g}^* \subset \mathfrak{d}$



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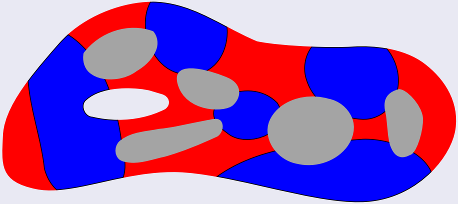

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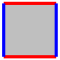
$H = Z($  $)$

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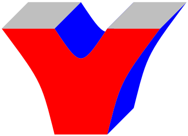

Thanks for your attention!