

Bosonic and fermionic T-dualization of type II superstring theory in double space

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Outline of the talk

- 1 T-duality
- 2 Model
- 3 Bosonic T-duality
- 4 Fermionic T-duality
- 5 Concluding remarks

Superstrings

- There are five consistent superstring theories. They are connected by web of T and S dualities.
- There are three approaches to superstring theory: NSR (Neveu-Schwarz-Ramond), GS (Green-Schwarz) and **pure spinor formalism** (N. Berkovits, hep-th/0001035).
- T-duality **transformation** does not change the physical content of the theory.
- Well known bosonic and recently discovered **fermionic** T-duality.

Idea of double space

- Double space = initial coordinates plus T-dual partners - Siegel, Duff, Tseytlin about 25 years ago.
- Interest for this subject emerged again (Hull, Berman, Zwiebach) in the context of T-duality as $O(d, d)$ transformation.
- The approach of Duff has been recently improved when the T-dualization along some subset of the initial and corresponding subset of the T-dual coordinates has been interpreted as permutation of these subsets in the double space coordinates ([arXiv:1505.06044](#), [1503.05580](#)). All calculations are made in full double space.
- In double space T-duality is a **symmetry** transformation.

General pure spinor action for type II superstring

- We start from the general pure spinor action for type II superstring (arXiv: 0405072)

$$\begin{aligned}
 S = & \int d^2\xi \left[\partial_+ \theta^\alpha A_{\alpha\beta} \partial_- \bar{\theta}^\beta + \partial_+ \theta^\alpha A_{\alpha\mu} \Pi_-^\mu + \Pi_+^\mu A_{\mu\alpha} \partial_- \bar{\theta}^\alpha \right. \\
 & + \Pi_+^\mu A_{\mu\nu} \Pi_-^\nu + d_\alpha E^\alpha{}_\beta \partial_- \bar{\theta}^\beta + d_\alpha E^\alpha{}_\mu \Pi_-^\mu + \partial_+ \theta^\alpha E_\alpha{}^\beta \bar{d}_\beta + \Pi_+^\mu E_\mu{}^\beta \bar{d}_\beta \\
 & + d_\alpha P^{\alpha\beta} \bar{d}_\beta + \frac{1}{2} N_+^{\mu\nu} \Omega_{\mu\nu,\beta} \partial_- \bar{\theta}^\beta + \frac{1}{2} N_+^{\mu\nu} \Omega_{\mu\nu,\rho} \Pi_-^\rho + \frac{1}{2} \partial_+ \theta^\alpha \Omega_{\alpha,\mu\nu} \bar{N}_-^{\mu\nu} \\
 & + \frac{1}{2} \Pi_+^\mu \Omega_{\mu,\nu\rho} \bar{N}_-^{\nu\rho} + \frac{1}{2} N_+^{\mu\nu} \bar{C}_{\mu\nu}{}^\beta \bar{d}_\beta + \frac{1}{2} d_\alpha C^\alpha{}_{\mu\nu} \bar{N}_-^{\mu\nu} \\
 & \left. + \frac{1}{4} N_+^{\mu\nu} S_{\mu\nu,\rho\sigma} \bar{N}_-^{\rho\sigma} \right] + S_\lambda + S_{\bar{\lambda}}. \tag{1}
 \end{aligned}$$

Bosonic T-duality - assumptions and approximations

- **Bosonic T-dualization** - we assume that background fields are independent of x^μ . In mentioned reference, expressions for background fields as well as action are obtained in an iterative procedure as an expansion in powers of θ^α and $\bar{\theta}^\alpha$. Every step in iterative procedure depends on previous one, so, for mathematical simplicity, we consider only basic (θ and $\bar{\theta}$ independent) components.

Fermionic T-duality - assumptions and consistency check

- **Fermionic T-dualization** - we assume that θ^α and $\bar{\theta}^\alpha$ are Killing directions. Consequently, auxiliary superfields are zero according to arXiv: 0405072. If we assume that rest of background fields are constant then their curvatures are zero. Using space-time field equations we confirmed the consistency of the choice of constant $P^{\alpha\beta}$.

Action

- In both cases, under introduced assumptions, action gets the form

$$\begin{aligned}
 S = & \kappa \int_{\Sigma} d^2\xi \left[\partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu + \frac{1}{4\pi\kappa} \Phi R^{(2)} \right] \\
 & + \int_{\Sigma} d^2\xi \left[-\pi_\alpha \partial_- (\theta^\alpha + \Psi_\mu^\alpha x^\mu) + \partial_+ (\bar{\theta}^\alpha + \bar{\Psi}_\mu^\alpha x^\mu) \bar{\pi}_\alpha + \frac{e^{\frac{\Phi}{2}}}{2\kappa} \pi_\alpha F^{\alpha\beta} \bar{\pi}_\beta \right]
 \end{aligned} \tag{2}$$

- Definitions: $\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}$, Φ is dilaton field, Ψ_μ^α and $\bar{\Psi}_\mu^\alpha$ are NS-R fields and $F^{\alpha\beta}$ is R-R field strength. Momenta π_α and $\bar{\pi}_\alpha$ are canonically conjugated to θ^α and $\bar{\theta}^\alpha$. All spinors are Majorana-Weyl ones.
- All background fields are constant.

Busher bosonic T-duality

- Global shift symmetry exists $x^a \rightarrow x^a + b$, where index a is subset of μ .
- We introduce gauge fields v_{\pm}^a and make change in the action $\partial_{\pm} x^a \rightarrow \partial_{\pm} x^a + v_{\pm}^a$.
- Additional term in the action

$$S_{gauge}(y, v_{\pm}) = \frac{1}{2} \kappa \int_{\Sigma} d^2 \xi (v_{+}^a \partial_{-} y_a - \partial_{+} y_a v_{-}^a),$$

where y_a is Lagrange multiplier. It makes v_{\pm}^a to be unphysical degrees of freedom.

- On the equations of motion for y_a we get initial action, while, fixing x^a to zero, on the equations of motion for v_{\pm}^a we get T-dual action.

Transformation laws

- Solution of the equation of motion for y_a is $v_{\pm}^a = \partial_{\pm} x^a$.
 Combining this solution with equations of motion for gauge fields v_{\pm}^a we obtain T-dual transformation laws

$$\partial_{\pm} x^a \cong -2\kappa \hat{\theta}_{\pm}^{ab} \Pi_{\mp bi} \partial_{\pm} x^i - \kappa \hat{\theta}_{\pm}^{ab} (\partial_{\pm} y_b - J_{\pm b}), \quad (3)$$

$$\partial_{\pm} y_a \cong -2\Pi_{\mp ab} \partial_{\pm} x^b - 2\Pi_{\mp ai} \partial_{\pm} x^i + J_{\pm a}. \quad (4)$$

Here $J_{\pm\mu} = \pm \frac{2}{\kappa} \Psi_{\pm\mu}^{\alpha} \pi_{\pm\alpha}$ and $\theta_{\pm}^{ac} \Pi_{\mp cb} = \frac{1}{2\kappa} \delta^a_b$, where

$$\Psi_{+\mu}^{\alpha} \equiv \Psi_{\mu}^{\alpha}, \quad \Psi_{-\mu}^{\alpha} \equiv \bar{\Psi}_{\mu}^{\alpha}, \quad \pi_{+\alpha} \equiv \pi_{\alpha}, \quad \pi_{-\alpha} \equiv \bar{\pi}_{\alpha}. \quad (5)$$

Transformation laws in double space

- In double space spanned by $Z^M = (x^\mu, y_\mu)^T$ they are of the form

$$\partial_\pm Z^M \cong \pm \Omega^{MN} \left(\mathcal{H}_{NP} \partial_\pm Z^P + J_{\pm N} \right), \quad (6)$$

where

$$\mathcal{H}_{MN} = \begin{pmatrix} G_{\mu\nu}^E & -2 B_{\mu\rho} (G^{-1})^{\rho\nu} \\ 2 (G^{-1})^{\mu\rho} B_{\rho\nu} & (G^{-1})^{\mu\nu} \end{pmatrix}, \quad (7)$$

is so called generalized metric, while

$$\Omega^{MN} = \begin{pmatrix} 0 & 1_D \\ 1_D & 0 \end{pmatrix}, J_{\pm M} = \begin{pmatrix} 2 (\Pi_\pm G^{-1})_\mu^\nu J_{\pm\nu} \\ - (G^{-1})^{\mu\nu} J_{\pm\nu} \end{pmatrix}. \quad (8)$$

Ω^{MN} is constant symmetric matrix and it is known as $SO(D, D)$ invariant metric. Here $G_{\mu\nu}^E = G_{\mu\nu} - 4 (B G^{-1} B)_{\mu\nu}$.

T-duality as permutation in double space

- T-dualization in double space is represented by permutation

$${}_a Z^M \equiv \begin{pmatrix} y_a \\ x^i \\ x^a \\ y_i \end{pmatrix} = (\mathcal{T}^a)^M{}_N Z^N \equiv \begin{pmatrix} 0 & 0 & 1_a & 0 \\ 0 & 1_j & 0 & 0 \\ 1_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_j \end{pmatrix} \begin{pmatrix} x^a \\ x^i \\ y_a \\ y_i \end{pmatrix} .$$

T-duality as permutation in double space

- Demanding that ${}_a Z^M$ has the transformation law as initial coordinates Z^M , we find the T-dual generalized metric

$${}_a \mathcal{H}_{MN} = (\mathcal{T}^a)_M{}^K \mathcal{H}_{KL} (\mathcal{T}^a)^L{}_N, \quad (9)$$

and T-dual current

$${}_a J_{\pm M} = (\mathcal{T}^a)_M{}^N J_{\pm N}. \quad (10)$$

NS-NS background fields

- From (9) we obtain the T-dual NS-NS background fields which are in full agreement with those obtained by Buscher procedure

$$\begin{aligned}
 {}_a\Pi_{\pm}^{ab} &= \frac{\kappa}{2}\hat{\theta}_{\mp}^{ab}, & {}_a\Pi_{\pm i}^a &= \kappa\hat{\theta}_{\mp}^{ab}\Pi_{\pm bi}, \\
 {}_a\Pi_{\pm i}^a &= -\kappa\Pi_{\pm ib}\hat{\theta}_{\mp}^{ba}, & {}_a\Pi_{\pm ij} &= \Pi_{\pm ij} - 2\kappa\Pi_{\pm ia}\hat{\theta}_{\mp}^{ab}\Pi_{\pm bj}.
 \end{aligned}$$

NS-R background fields

- From (10) we obtain the form of the T-dual NS-R fields

$${}_a\Psi^{\alpha a} = \kappa \hat{\theta}_+^{ab} \Psi_b^\alpha, \quad {}_a\bar{\Psi}^{\alpha a} = \kappa {}_a\Omega^\alpha{}_\beta \hat{\theta}_-^{ab} \bar{\Psi}_b^\beta. \quad (11)$$

$${}_a\Psi_i^\alpha = \Psi_i^\alpha - 2\kappa \Pi_{-ib} \hat{\theta}_+^{ba} \Psi_a^\alpha, \quad {}_a\bar{\Psi}_i^\alpha = {}_a\Omega^\alpha{}_\beta (\bar{\Psi}_i^\beta - 2\kappa \Pi_{+ib} \hat{\theta}_-^{ba} \bar{\Psi}_a^\beta). \quad (12)$$

- From transformation laws we see that two chiral sectors transform differently. Consequently, there are two sets of vielbeins in T-dual picture as well two sets of gamma matrices. This T-dual vielbeins are connected by Lorentz transformation, while spinorial representation of this Lorentz transformation, ${}_a\Omega^\alpha{}_\beta$, relates two sets of gamma matrices. In order to have unique set of gamma matrices, we have to multiply one fermionic index by ${}_a\Omega^\alpha{}_\beta$.

R-R field strength

- R-R field strength couples fermionic momenta and, consequently, its T-dual can not be read from transformation law.
- From the demand that term in the action is T-dual invariant, we obtain the form of the T-dual R-R field strength

$$e^{\frac{a\Phi}{2}} {}_a F^{\alpha\beta} = (e^{\frac{\Phi}{2}} F^{\alpha\gamma} + c \Psi_a^\alpha \hat{\theta}^{ab} \bar{\Psi}_b^\gamma) {}_a \Omega_\gamma^\beta, \quad (13)$$

where c is an arbitrary constant. For the specific value of c , we get the same expression as in Buscher procedure.

Basic facts

- In last years it was seen that tree level superstring theories on certain supersymmetric backgrounds admit a symmetry which is called fermionic T-duality.
- This is a redefinition of the fermionic worldsheet fields similar to the redefinition we perform on bosonic variables when we do an ordinary T-duality.
- Technically, the procedure is the same as in the bosonic case up to the fact that dualization will be done along θ^α and $\bar{\theta}^\alpha$ directions.

Action

- On the equations of motion for π_α and $\bar{\pi}_\alpha$ action (2) becomes

$$\begin{aligned}
 \mathcal{S} = & \kappa \int_{\Sigma} d^2\xi \partial_+ x^\mu \left[\Pi_{+\mu\nu} + \frac{1}{2} \bar{\Psi}_\mu^\alpha (P^{-1})_{\alpha\beta} \Psi_\nu^\beta \right] \partial_- x^\nu \\
 + & \frac{1}{4\pi} \int_{\Sigma} d^2\xi \Phi R^{(2)} \\
 + & \frac{\kappa}{2} \int_{\Sigma} d^2\xi \left[\partial_+ \bar{\theta}^\alpha (P^{-1})_{\alpha\beta} \partial_- \theta^\beta + \partial_+ \bar{\theta}^\alpha (P^{-1} \Psi)_{\alpha\mu} \partial_- x^\mu \right. \\
 + & \left. \partial_+ x^\mu (\bar{\Psi} P^{-1})_{\mu\alpha} \partial_- \theta^\alpha \right].
 \end{aligned}$$

Fixing the chiral gauge invariance

- In the above action θ^α appears only in the form $\partial_- \theta^\alpha$ and $\bar{\theta}^\alpha$ in the form $\partial_+ \bar{\theta}^\alpha$.
- Using the BRST formalism we fix this chiral gauge invariance adding to the action

$$S_{gf} = -\frac{\kappa}{2} \int d^2\xi \partial_- \bar{\theta}^\alpha (\alpha^{-1})_{\alpha\beta} \partial_+ \theta^\beta, \quad (14)$$

where $\alpha^{\alpha\beta}$ is arbitrary non singular matrix.

Transformation laws

- Applying the same mathematical procedure as in the case of the bosonic T-dualization, we have

$$\partial_- \theta^\alpha \cong -P^{\alpha\beta} \partial_- \vartheta_\beta - \Psi_\mu^\alpha \partial_- x^\mu, \quad \partial_+ \bar{\theta}^\alpha \cong \partial_+ \bar{\vartheta}_\beta P^{\beta\alpha} - \partial_+ x^\mu \bar{\Psi}_\mu^\alpha, \quad (15)$$

$$\partial_+ \theta^\alpha \cong -\alpha^{\alpha\beta} \partial_+ \vartheta_\beta, \quad \partial_- \bar{\theta}^\alpha \cong \partial_- \bar{\vartheta}_\beta \alpha^{\beta\alpha}, \quad (16)$$

where ϑ_α and $\bar{\vartheta}_\alpha$ are T-dual fermionic coordinates.

Transformation laws in double space

- Let us introduce double fermionic coordinates

$$\Theta^A = \begin{pmatrix} \theta^\alpha \\ \vartheta_\alpha \end{pmatrix}, \quad \bar{\Theta}^A = \begin{pmatrix} \bar{\theta}^\alpha \\ \bar{\vartheta}_\alpha \end{pmatrix}. \quad (17)$$

- Transformation laws in double space are of the form

$$\partial_- \Theta^A \cong -\Omega^{AB} \left[\mathcal{F}_{BC} \partial_- \Theta^C + \mathcal{J}_{-B} \right],$$

$$\partial_+ \bar{\Theta}^A \cong \left[\partial_+ \bar{\Theta}^C \mathcal{F}_{CB} + \bar{\mathcal{J}}_{+B} \right] \Omega^{BA},$$

$$\partial_+ \Theta^A \cong -\Omega^{AB} \mathcal{A}_{BC} \partial_+ \Theta^C, \quad \partial_- \bar{\Theta}^A \cong \partial_- \bar{\Theta}^C \mathcal{A}_{CB} \Omega^{BA}.$$

Generalized metric and currents

- The generalized metric and the matrix \mathcal{A}_{AB} are

$$\mathcal{F}_{AB} = \begin{pmatrix} (P^{-1})_{\alpha\beta} & 0 \\ 0 & P^{\gamma\delta} \end{pmatrix}, \mathcal{A}_{AB} = \begin{pmatrix} (\alpha^{-1})_{\alpha\beta} & 0 \\ 0 & \alpha^{\gamma\delta} \end{pmatrix}.$$

- The currents are of the form

$$\bar{\mathcal{J}}_{+A} = \begin{pmatrix} (\bar{\Psi} P^{-1})_{\mu\alpha} \partial_+ X^\mu \\ -\bar{\Psi}^\alpha_\mu \partial_+ X^\mu \end{pmatrix}, \mathcal{J}_{-A} = \begin{pmatrix} (P^{-1} \Psi)_{\alpha\mu} \partial_- X^\mu \\ \Psi^\alpha_\mu \partial_- X^\mu \end{pmatrix}.$$

Fermionic T-dualization as permutation

- T-dual coordinates are

$$*\Theta^A = \mathcal{T}^A_B \Theta^B, \quad *\bar{\Theta}^A = \mathcal{T}^A_B \bar{\Theta}^B,$$

where

$$\mathcal{T}^A_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

is permutation matrix.

Fermionic T-dualization as permutation

- Demanding that T-dual coordinates transformation laws are of the same form as those for initial coordinates we get

$${}^* \mathcal{F}_{AB} = \mathcal{T}_A^C \mathcal{F}_{CD} \mathcal{T}_B^D, \quad {}^* \bar{\mathcal{J}}_{+A} = \mathcal{T}_A^B \bar{\mathcal{J}}_{+B}, \quad {}^* \mathcal{J}_{-A} = \mathcal{T}_A^B \mathcal{J}_{-B}.$$

- The matrix \mathcal{A}_{AB} transforms as

$${}^* \mathcal{A}_{AB} = \mathcal{T}_A^C \mathcal{A}_{CD} \mathcal{T}_B^D = (\mathcal{A}^{-1})_{AB}. \quad (18)$$

Background fields

- From these relations we obtain the R-R and NS-R T-dual background fields in the same form as in the Buscher procedure

$$\begin{aligned}
 {}^*P_{\alpha\beta} &= (P^{-1})_{\alpha\beta}, & ({}^*\alpha)_{\alpha\beta} &= (\alpha^{-1})_{\alpha\beta}, \\
 {}^*\Psi_{\alpha\mu} &= (P^{-1})_{\alpha\beta} \Psi_{\mu}^{\beta}, & {}^*\bar{\Psi}_{\alpha\mu} &= -\bar{\Psi}_{\mu}^{\beta} (P^{-1})_{\beta\alpha}.
 \end{aligned}$$

NS-NS background fields

- $\Pi_{+\mu\nu}$ is coupled by x 's and we can not read the T-dual field from transformation laws.
- As in the case of bosonic T-dualization, assuming that this term is invariant under T-dualization, we get the appropriate fermionic T-dual

$$*\Pi_{+\mu\nu} = \Pi_{+\mu\nu} + c\bar{\Psi}_{\mu}^{\alpha}(P^{-1})_{\alpha\beta}\Psi_{\nu}^{\beta}, \quad (19)$$

where c is an arbitrary constant.

Concluding remarks

- We represented both kind of T-dualizations of type II superstring as permutation symmetry in double space.
- The successive T-dualizations make a group called T-duality group. In the case of type II superstring fermionic T-duality transformations are performed by the same matrices \mathcal{T}^a as in the bosonic string case. Consequently, the corresponding T-duality group is the same.
- In the bosonic case there is an advantage of this approach. In one equation all T-dual theories (for any subset x^a) are contained. We do not have to repeat procedure for each specific choice of x^a . This kind of approach could be helpful in better understanding of M-theory.