Quantum field theories on quantum spaces

Jean-Christophe Wallet

Laboratoire de Physique Théorique d'Orsay,
CNRS and University of Paris-Sud 11

18-22 June 2018 / Primošten

About the title

- Quantum space = "noncommutative space" modeled by noncommutative algebra (with involution), says $\mathbb{A}$
  Examples: fuzzy spheres, NC torus, quantum spheres,...,
  deformations of $\mathbb{R}^n$ (Moyal, $\mathbb{R}^3_\lambda$,...), κ-deformations,...
  $\rightarrow$ We will focus on popular: Moyal spaces, $\mathbb{R}^3_\lambda$, κ-Minkowski
- Quantum field theory : Field theories = Noncommutative Field Theories (NCFT) built on $\mathbb{A}$,
  $\rightarrow$ We will focus on renormalisation aspects
- Huge literature on the subject
  Arising in various areas motivated by: String physics, Branes, quantum gravity approaches,...

About the title

- Quantum space = "noncommutative space" modeled by noncommutative algebra (with involution), says $\mathbb{A}$
  Examples: fuzzy spheres, NC torus, quantum spheres,...,
  deformations of $\mathbb{R}^n$ (Moyal, $\mathbb{R}^3_\lambda$,...), κ-deformations,...
  $\rightarrow$ We will focus on popular: Moyal spaces, $\mathbb{R}^3_\lambda$, κ-Minkowski
- Quantum field theory : Field theories = Noncommutative Field Theories (NCFT) built on $\mathbb{A}$,
  $\rightarrow$ We will focus on renormalisation aspects
- Huge literature on the subject
  Arising in various areas motivated by: String physics, Branes, quantum gravity approaches,...

About the title

- Quantum space = "noncommutative space" modeled by noncommutative algebra (with involution), says $\mathbb{A}$
  Examples: fuzzy spheres, NC torus, quantum spheres,...,
  deformations of $\mathbb{R}^n$ (Moyal, $\mathbb{R}^3_\lambda$,...), κ-deformations,...
  $\rightarrow$ We will focus on popular: Moyal spaces, $\mathbb{R}^3_\lambda$, κ-Minkowski
- Quantum field theory : Field theories = Noncommutative Field Theories (NCFT) built on $\mathbb{A}$,
  $\rightarrow$ We will focus on renormalisation aspects
- Huge literature on the subject
  Arising in various areas motivated by: String physics, Branes, quantum gravity approaches,...

Outline

1. NCFT on Moyal spaces and $\mathbb{R}^3_\lambda$ - present situation
   - Renormalisable NCFT on Moyal spaces
   - Renormalisable NCFT on $\mathbb{R}^3_\lambda$
   - (Twisted) convolution product and Moyal product
   - Summary
2. Star product on κ-Minkowski space from Weyl quantization
   - Weyl quantization and star product for κ-Minkowski space
   - Properties of the star product
   - Trading cyclicity for KMS condition
   - Family of scalar NCFT: 2-and 4-point functions at 1-loop
   - Conclusions

Renormalisable NCFT on Moyal spaces

Informally $\mathbb{C}[x_\mu]/[x_\mu, x_\nu]_\star = i\theta_{\mu\nu}$
$(\mathbb{F}(\mathbb{R}^{2n}), \star)$, $\star$: Moyal product. Few renormalisable NCFT
– Harmonic ϕ^4 model [Grosse, Wulkenhaar,CMP.**256**(2005)305]

$$S = \int d^4x \, \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{m^2}{2}\phi^2 + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}_\mu\phi) + \frac{\lambda}{4!}\,\phi\star\phi\star\phi\star\phi$$

Harmonic term cures UV/IR mixing (breaks translation invariance)
$\beta_\lambda = 0$ (all orders) for $\Omega = 1$ [Dissertori et al., Phys. Lett. B.**649** (2007) 95]
$\Omega = 1$ LSZ duality [Langmann, Szabo, Zarembo, JHEP 0401 (2004) 017]
– Translational-invariant model: (tree level) counterterms cancels IR singularity inducing mixing [Magnen, Rivasseau, Tanasa, CMP **287** (2009) 275]
– Rotationaly invariant models: use action of rotation on sympletic structure [de Goursac, JCW, J. Phys. A: Math. Theor. 44 (2011) 055401]

Renormalisable gauge theories on Moyal spaces?

– D=4 . Still UV/IR mixing. Gauge invariance forbids harmonic term
– Existence of some all order renormalisable gauge theory unknown
– Candidate proposed

[de Goursac, Wulkenhaar, JCW, EPJC**51** (2007) 977]

[Grosse, Wohlgenannt, EPJC**52** (2007)435]

$$S(A) = \int d^4x \Big(\frac{1}{4} F_{\mu\nu} \star F_{\mu\nu} + \frac{\Omega^2}{4} \{\mathcal{A}_\mu, \mathcal{A}_\nu\}^2_\star + \mu^2 \mathcal{A}_\mu \star \mathcal{A}_\mu \Big)$$

$$\mathcal{A}_\mu = A_\mu + \tfrac{1}{2}\tilde{x}_\mu$$
$$F_{\mu\nu} = -i[\mathcal{A}_\mu, \mathcal{A}_\nu]_\star + \theta^{-1}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star$$

$S(A)$ complicated vacuum structure
Direct perturbative treatment problematic

[de Goursac, Wulkenhaar, JCW, EPJC **56** (2008) 293]

Renormalisable gauge theories on Moyal spaces?

– Interpretation as matrix model using $\mathcal{A}_\mu$ interesting
Steinacker, Nucl. Phys. B**679** (2004) 66; Class. Quant. Grav. **27** (2010) 133001, etc...

– $S(\mathcal{A})$ $D = 2$ expanded around one particular vacuum investigated only at one loop order [Martinetti, Vitale, JCW, JHEP 09 (2013) 051]. Looks like but $\neq$ 6-vertex model. Vacuum instability.

NCFT on $\mathbb{R}^3_\lambda$

- Usual presentation: $\mathbb{C}[(x_i, x_0)]/\mathcal{I}$, $i = 1, 2, 3$

$$\mathcal{I} : [x_i, x_j]_\star = i\lambda\epsilon_{ijk}x_k,\ \ x_0^2 - \lambda x_0 = \sum_i x_i^2$$

  Deformed algebras of function $\mathbb{R}^3_\lambda = (\mathbb{F}(\mathbb{R}^3), \star)$: complicated star products not convenient for perturbative computations.
  Various star products available:

- Gracia-Bondia et al., JHEP 0204 (2002) 026; Freidel, Livine, PRL **96** (2006) 221301; Freidel, Majid, Class. Quant. Grav.**25** (2008)045006; Kupriyanov, Vassilevich, EPJC**58** (2008)627; Jurić, Poulain, JCW, JHEP 1707 (2017) 116.

NCFT on $\mathbb{R}^3_\lambda$

- Usual presentation: $\mathbb{C}[(x_i, x_0)]/\mathcal{I}$, $i = 1, 2, 3$

$$\mathcal{I} : [x_i, x_j]_\star = i\lambda\epsilon_{ijk}x_k,\ x_0^2 - \lambda x_0 = \sum_i x_i^2$$

  Deformed algebras of function $\mathbb{R}^3_\lambda = (\mathbb{F}(\mathbb{R}^3), \star)$: complicated star products not convenient for perturbative computations.
  Various star products available:

- Gracia-Bondia et al., JHEP 0204 (2002) 026; Freidel, Livine, PRL **96** (2006) 221301; Freidel, Majid, Class. Quant. Grav.**25** (2008)045006; Kupriyanov, Vassilevich, EPJC**58** (2008)627; Jurić, Poulain, JCW, JHEP 1707 (2017) 116.

Renormalisable NCFT on $\mathbb{R}^3_\lambda$

Conveniently described as $\mathbb{R}^3_\lambda \to (\hat{\mathbb{R}}^3_\lambda = \oplus_{j\in\frac{\mathbb{N}}{2}} \mathbb{M}_{2j+1}(\mathbb{C}), .)$
$\hat{f} = \oplus_j \int d\mu(x) f(x) t^j(x) \longrightarrow$ yields $\hat{f} = \sum_j a^j_{m,n} \hat{v}^j_{mn}$

Characterize the natural "matrix base" used to represent NCFT on $\mathbb{R}^3_\lambda$ as matrix models [Vitale, JCW, JHEP 1304 (2013) 115]
– One-loop calculations:
• Scalar ϕ^4 models (toy models): UV softer than commutative case
Sometimes UV/IR mixing. All order renormalisability?
[Vitale, JCW, JHEP 1304 (2013) 115].
[see also Jurić, Poulain, JCW JHEP 05 (2016) 146]

• Gauge theories $S(A)$ (massless) from simple noncommutative differential calculus. Vacuum instability!
[Géré, Vitale, JCW, Phys. Rev. D**90**,(2014) 045019].
When truncated to one $\mathbb{M}_{2j+1}(\mathbb{C})$, almost similar to brane model
[Alekseev et al., JHEP 05 (2000) 010]

Finite gauge theories on $\mathbb{R}^3_\lambda$ [Géré, Jurić, JCW, JHEP 12 (2015) 045]

$$\theta_\mu = \frac{x_\mu}{\lambda^2} \ \rightarrow \ \mathcal{A}_\mu = A_\mu + i\theta_\mu, \ F_{\mu\nu} = [\mathcal{A}_\mu, \mathcal{A}_\nu] + \frac{1}{\lambda}\varepsilon_{\mu\nu\rho}\mathcal{A}_\mu$$

- Set $\Phi_\mu := \mathcal{A}_\mu$. $\Phi_\mu^g = g^\dagger \Phi_\mu g$. Observe: $(\theta_\mu\theta^\mu) \in \mathcal{Z}(\mathbb{R}^3_\lambda)$
- $Tr((\theta_\mu\theta^\mu)\Phi_\nu\Phi^\nu)$ gauge invariant. Harmonic term now allowed!
- One obtains (in the gauge $\Phi_3 = \theta_3$; $\Phi = \Phi_1 + i\Phi_2$)

$$S = \frac{2}{g^2} Tr(\Phi Q \Phi^\dagger + \Phi^\dagger Q \Phi) + \frac{16}{g^2} Tr((\Omega+1)\Phi\Phi^\dagger\Phi\Phi^\dagger + (3\Omega-1)\Phi\Phi^\dagger\Phi^\dagger\Phi)$$

$$Q = M + \mu x^2 + 8\Omega L(\theta_3^2) + i4(\Omega - 1)L(\theta_3)D_3$$

S positive for $M > 0$, $\mu > 0$, $\Omega \in [0, \frac{4}{3}]$.

Theorem (Géré, Jurić, JCW, JHEP 12 (2015) 045)

The amplitudes of the (ribbon) diagrams for any of the gauge theories described by $S(\Phi)$ with $M > 0$, $\mu > 0$, $\Omega \in [0, \frac{4}{3}]$, are finite to all orders in perturbation.

Finite gauge theories on $\mathbb{R}^3_\lambda$ [Géré, Jurić, JCW, JHEP 12 (2015) 045]

$$\theta_\mu = \frac{x_\mu}{\lambda^2} \ \to \ \mathcal{A}_\mu = A_\mu + i\theta_\mu, \ F_{\mu\nu} = [\mathcal{A}_\mu, \mathcal{A}_\nu] + \frac{1}{\lambda}\varepsilon_{\mu\nu\rho}\mathcal{A}_\mu$$

- Set $\Phi_\mu := \mathcal{A}_\mu$. $\Phi^g_\mu = g^\dagger \Phi_\mu g$. Observe: $(\theta_\mu\theta^\mu) \in \mathcal{Z}(\mathbb{R}^3_\lambda)$
- $Tr\big((\theta_\mu\theta^\mu)\Phi_\nu\Phi^\nu\big)$ gauge invariant. Harmonic term now allowed!
- One obtains (in the gauge $\Phi_3 = \theta_3$; $\Phi = \Phi_1 + i\Phi_2$)

$$S = \frac{2}{g^2} Tr(\Phi Q\Phi^\dagger + \Phi^\dagger Q\Phi) + \frac{16}{g^2} Tr\big((\Omega+1)\Phi\Phi^\dagger\Phi\Phi^\dagger + (3\Omega-1)\Phi\Phi^\dagger\Phi^\dagger\Phi\big)$$

$$Q = M + \mu x^2 + 8\Omega L(\theta_3^2) + i4(\Omega-1)L(\theta_3)D_3$$

S positive for $M > 0$, $\mu > 0$, $\Omega \in [0, \frac{4}{3}]$.

Theorem (Géré, Jurić, JCW, JHEP 12 (2015) 045)

The amplitudes of the (ribbon) diagrams for any of the gauge theories described by $S(\Phi)$ with $M > 0$, $\mu > 0$, $\Omega \in [0, \frac{4}{3}]$, are finite to all orders in perturbation.

Finite gauge theories on $\mathbb{R}^3_\lambda$ [Géré, Jurić, JCW, JHEP 12 (2015) 045]

$$\theta_\mu = \frac{x_\mu}{\lambda^2} \;\to\; \mathcal{A}_\mu = A_\mu + i\theta_\mu,\; F_{\mu\nu} = [\mathcal{A}_\mu, \mathcal{A}_\nu] + \frac{1}{\lambda}\varepsilon_{\mu\nu\rho}\mathcal{A}_\mu$$

- Set $\Phi_\mu := \mathcal{A}_\mu$. $\Phi^g_\mu = g^\dagger \Phi_\mu g$. Observe: $(\theta_\mu\theta^\mu) \in \mathcal{Z}(\mathbb{R}^3_\lambda)$
- $Tr\big((\theta_\mu\theta^\mu)\Phi_\nu\Phi^\nu\big)$ gauge invariant. Harmonic term now allowed!
- One obtains (in the gauge $\Phi_3 = \theta_3$; $\Phi = \Phi_1 + i\Phi_2$)

$$S = \frac{2}{g^2} Tr\big(\Phi Q \Phi^\dagger + \Phi^\dagger Q \Phi\big) + \frac{16}{g^2} Tr\big((\Omega+1)\Phi\Phi^\dagger\Phi\Phi^\dagger + (3\Omega-1)\Phi\Phi^\dagger\Phi^\dagger\Phi\big)$$

$$Q = M + \mu x^2 + 8\Omega L(\theta_3^2) + i4(\Omega-1)L(\theta_3)D_3$$

S positive for $M > 0$, $\mu > 0$, $\Omega \in [0, \frac{4}{3}]$.

Theorem (Géré, Jurić, JCW, JHEP 12 (2015) 045)

The amplitudes of the (ribbon) diagrams for any of the gauge theories described by $S(\Phi)$ with $M > 0$, $\mu > 0$, $\Omega \in [0, \frac{4}{3}]$, are finite to all orders in perturbation.

Finite gauge theories on $\mathbb{R}^3_\lambda$ [Géré, Jurić, JCW, JHEP 12 (2015) 045]

$$\theta_\mu = \frac{x_\mu}{\lambda^2} \ \rightarrow \ \mathcal{A}_\mu = A_\mu + i\theta_\mu, \ F_{\mu\nu} = [\mathcal{A}_\mu, \mathcal{A}_\nu] + \frac{1}{\lambda}\varepsilon_{\mu\nu\rho}\mathcal{A}_\mu$$

- Set $\Phi_\mu := \mathcal{A}_\mu$. $\Phi^g_\mu = g^\dagger \Phi_\mu g$. Observe: $(\theta_\mu\theta^\mu) \in \mathcal{Z}(\mathbb{R}^3_\lambda)$
- $Tr\big((\theta_\mu\theta^\mu)\Phi_\nu\Phi^\nu\big)$ gauge invariant. Harmonic term now allowed!
- One obtains (in the gauge $\Phi_3 = \theta_3$; $\Phi = \Phi_1 + i\Phi_2$)

$$S = \frac{2}{g^2} Tr\big(\Phi Q \Phi^\dagger + \Phi^\dagger Q \Phi\big) + \frac{16}{g^2} Tr\big((\Omega+1)\Phi\Phi^\dagger\Phi\Phi^\dagger + (3\Omega-1)\Phi\Phi^\dagger\Phi^\dagger\Phi\big)$$

$$Q = M + \mu x^2 + 8\Omega L(\theta_3^2) + i4(\Omega-1)L(\theta_3)D_3$$

S positive for $M > 0$, $\mu > 0$, $\Omega \in [0, \frac{4}{3}]$.

Theorem (Géré, Jurić, JCW, JHEP 12 (2015) 045)

The amplitudes of the (ribbon) diagrams for any of the gauge theories described by $S(\Phi)$ with $M > 0$, $\mu > 0$, $\Omega \in [0, \frac{4}{3}]$, are finite to all orders in perturbation.

Finite gauge theories on $\mathbb{R}^3_\lambda$

– *S* describes dynamics of fluctuations of Φ_μ around $\Phi_\mu = 0$ or alternatively fluctuations of the gauge potential A_μ around θ_μ
– Finiteness origin:

- Sufficient decay for propagator as $j \to \infty$ (UV regime)
- *j* plays role of natural UV cut-off (kind of "external moment")
- Existence of (finite) upper bound for general amplitude $\mathfrak{A}^j_{\mathcal{D}}$

– Commutative limit: does not reproduce Yang-Mills theorie.

– Solvable for $\Omega = \frac{1}{3}$ [JCW, Nucl. Phys. B912 (2016) 354]

Finite gauge theories on $\mathbb{R}^3_\lambda$

– *S* describes dynamics of fluctuations of Φ_μ around $\Phi_\mu = 0$ or alternatively fluctuations of the gauge potential A_μ around θ_μ
– Finiteness origin:

- Sufficient decay for propagator as $j \to \infty$ (UV regime)
- j plays role of natural UV cut-off (kind of "external moment")
- Existence of (finite) upper bound for general amplitude $\mathfrak{A}^j_{\mathcal{D}}$

– Commutative limit: does not reproduce Yang-Mills theorie.

– Solvable for $\Omega = \frac{1}{3}$ [JCW, Nucl. Phys. B912 (2016) 354]

Finite gauge theories on $\mathbb{R}^3_\lambda$

– *S* describes dynamics of fluctuations of Φ_μ around $\Phi_\mu = 0$ or alternatively fluctuations of the gauge potential A_μ around θ_μ

– Finiteness origin:

- Sufficient decay for propagator as $j \to \infty$ (UV regime)
- j plays role of natural UV cut-off (kind of "external moment")
- Existence of (finite) upper bound for general amplitude $\mathfrak{A}^j_{\mathcal{D}}$

– Commutative limit: does not reproduce Yang-Mills theorie.

– Solvable for $\Omega = \frac{1}{3}$ [JCW, Nucl. Phys. B912 (2016) 354]

Exactly solvable gauge theory on $\mathbb{R}^3_\lambda$

JCW, Nucl. Phys. B912 (2016) 354

For $\Omega = \frac{1}{3}$ in S above, partition function $Z(Q)$ exactly computable

$$Z(Q) = \prod_{j \in \frac{\mathbb{N}}{2}} Z_j(Q),\ Z_j(Q) \sim \frac{\det_{-j\leq m,n\leq j}\Big(f(\omega^j_m + \omega^j_n)\Big)}{\Delta^2(Q^j)} \quad ,$$

$$f(x) = \sqrt{\frac{\pi g^2}{128(j+1)}}\ \mathrm{erfc}\Big(x\sqrt{\frac{(j+1)}{64g^2}}\Big)\ \ e^{x\frac{(j+1)}{64g^2}}$$

$\Delta(Q^j)$ Vandermonde determinant, function of eigenvalues ω^j_n of kinetic operator Q. $Z_j(Q)$: τ-function for integrable 2-d Toda hierarchy.

(Twisted) convolution product and Moyal product

$$(f\star g)(x)=\int d^nk_2\ e^{ik_2x}\times\left[d^nk_1\mathcal{F}f(k_1)\mathcal{F}g(k_2-k_1)e^{\frac{i}{2}k_2\theta k_1})\right]$$

Fourier transform of $[...]$ = a twisted convolution product $\hat{\circ}$

– $\mathbb{H}$ Heisenberg group, $F(z,u)\in\mathbb{C}[\mathbb{H}]:=(L^1(\mathbb{H}),\circ)$, $u\in\mathbb{R}^{2n}$, $z\in\mathbb{R}$

– Twist: $F^{\#}(u):=\int dz\ F(z,u)e^{i\frac{\hbar}{2}z}$ Obtained from representation:

$\pi(F)(z,u)=\int d^{2n}udz\ F(z,u)\left[e^{\frac{\hbar}{2}z}U(u)\right]$, $[.]$ projective unireps of $\mathbb{R}^{2n}$

– Hence $(F\circ G)^{\#}(u)=(F^{\#}\hat{\circ}G^{\#})(u)$

– Set $F^{\#}(u)=\mathcal{F}f(u)$, i.e functions on $2n$-dim momentum space

– Weyl quantization map $W(f):=\pi(\mathcal{F}f)$

use $W(f\star g)=W(f)W(g)$, $\pi(F\circ G)^{\#}(u)=\pi(F^{\#})\pi(G^{\#})$ to get

$$f\star g=\mathcal{F}^{-1}(\mathcal{F}f\hat{\circ}\mathcal{F}g)$$

Summary

(Up to the twist)

- Start from group G related to coordinates algebra
- Consider $\mathbb{C}[G]=(L^1(G),\circ)$, its representation $\pi:\mathbb{C}[G]\to\mathcal{B}(\mathcal{H})$ $\pi(F)=\int_G d\nu(s)F(s)\pi_U(s)$,
- $F=\mathcal{F}f$, i.e interpreted as functions on momentum space
- Weyl-type quantization $Q(f)=\pi(\mathcal{F}f)$, Q *-morphism
- use $\pi(F\circ G)=\pi(F)\pi(G)$, $\pi(F)^\dagger=\pi(F^*)$ to get

$$
\begin{aligned}
f\star g &= \mathcal{F}^{-1}(\mathcal{F}f\circ\mathcal{F}g)\\
f^\dagger &= \mathcal{F}^{-1}(\mathcal{F}(f)^*
\end{aligned}
$$

see e.g Hennings, Dubin, Publ. RIMS, Kyoto Univ. 45 (2009), 1041

Extends to κ-Minkowski

[Durhuus, Sitarz, J. Noncom.Geom. 7 (2013) 605, Poulain, JCW, PRD (2018) in press, arXiv:1801.02715]

Summary

(Up to the twist)

- Start from group G related to coordinates algebra
- Consider $\mathbb{C}[G]=(L^1(G),\circ)$, its representation $\pi:\mathbb{C}[G]\to\mathcal{B}(\mathcal{H})$ $\pi(F)=\int_G d\nu(s)F(s)\pi_U(s)$,
- $F=\mathcal{F}f$, i.e interpreted as functions on momentum space
- Weyl-type quantization $Q(f)=\pi(\mathcal{F}f)$, Q *-morphism
- use $\pi(F\circ G)=\pi(F)\pi(G)$, $\pi(F)^\dagger=\pi(F^*)$ to get

$$\begin{aligned} f\star g &= \mathcal{F}^{-1}(\mathcal{F}f\circ\mathcal{F}g) \\ f^\dagger &= \mathcal{F}^{-1}(\mathcal{F}(f)^* \end{aligned}$$

see e.g Hennings, Dubin, Publ. RIMS, Kyoto Univ. 45 (2009), 1041

Extends to κ-Minkowski

[Durhuus, Sitarz, J. Noncom.Geom. 7 (2013) 605, Poulain, JCW, PRD (2018) in press, arXiv:1801.02715]

Summary

(Up to the twist)

- Start from group G related to coordinates algebra
- Consider $\mathbb{C}[G] = (L^1(G), \circ)$, its representation $\pi : \mathbb{C}[G] \to \mathcal{B}(\mathcal{H})$ $\pi(F) = \int_G d\nu(s) F(s) \pi_U(s)$,
- $F = \mathcal{F}f$, i.e interpreted as functions on momentum space
- Weyl-type quantization $Q(f) = \pi(\mathcal{F}f)$, Q *-morphism
- use $\pi(F \circ G) = \pi(F)\pi(G)$, $\pi(F)^\dagger = \pi(F^*)$ to get

$$\begin{aligned} f \star g &= \mathcal{F}^{-1}(\mathcal{F}f \circ \mathcal{F}g) \\ f^\dagger &= \mathcal{F}^{-1}(\mathcal{F}(f)^* \end{aligned}$$

see e.g Hennings, Dubin, Publ. RIMS, Kyoto Univ. 45 (2009), 1041

Extends to κ-Minkowski

[Durhuus, Sitarz, J. Noncom.Geom. 7 (2013) 605, Poulain, JCW, PRD (2018) in press, arXiv:1801.02715]

Summary

(Up to the twist)

- Start from group G related to coordinates algebra
- Consider $\mathbb{C}[G]=(L^1(G),\circ)$, its representation $\pi:\mathbb{C}[G]\to\mathcal{B}(\mathcal{H})$ $\pi(F)=\int_G d\nu(s)F(s)\pi_U(s)$,
- $F=\mathcal{F}f$, i.e interpreted as functions on momentum space
- Weyl-type quantization $Q(f)=\pi(\mathcal{F}f)$, Q *-morphism
- use $\pi(F\circ G)=\pi(F)\pi(G)$, $\pi(F)^\dagger=\pi(F^*)$ to get

$$\begin{aligned} f\star g &= \mathcal{F}^{-1}(\mathcal{F}f\circ\mathcal{F}g)\\ f^\dagger &= \mathcal{F}^{-1}(\mathcal{F}(f)^* \end{aligned}$$

see e.g Hennings, Dubin, Publ. RIMS, Kyoto Univ. 45 (2009), 1041

Extends to κ-Minkowski

[Durhuus, Sitarz, J. Noncom.Geom. 7 (2013) 605, Poulain, JCW, PRD (2018) in press, arXiv:1801.02715]

Summary

(Up to the twist)

- Start from group G related to coordinates algebra
- Consider $\mathbb{C}[G] = (L^1(G), \circ)$, its representation $\pi : \mathbb{C}[G] \to \mathcal{B}(\mathcal{H})$ $\pi(F) = \int_G d\nu(s) F(s) \pi_U(s)$,
- $F = \mathcal{F}f$, i.e interpreted as functions on momentum space
- Weyl-type quantization $Q(f) = \pi(\mathcal{F}f)$, Q *-morphism
- use $\pi(F \circ G) = \pi(F)\pi(G)$, $\pi(F)^\dagger = \pi(F^*)$ to get

$$\begin{aligned} f \star g &= \mathcal{F}^{-1}(\mathcal{F}f \circ \mathcal{F}g) \\ f^\dagger &= \mathcal{F}^{-1}(\mathcal{F}(f)^* \end{aligned}$$

see e.g Hennings, Dubin, Publ. RIMS, Kyoto Univ. 45 (2009), 1041

Extends to κ-Minkowski

[Durhuus, Sitarz, J. Noncom.Geom. 7 (2013) 605, Poulain, JCW, PRD (2018) in press, arXiv:1801.02715]

Convolution algebra for κ-Minkowski space

Another quantum space with "Lie algebra-type noncommutativity"

- Lie algebra $\mathfrak{g}$ ($\kappa > 0$): (solvable)

$$[x_0, x_i] = \frac{i}{\kappa} x_i, \quad [x_i, x_j] = 0, \quad i, j = 1, \cdots, d.$$

  Informally, κ-Minkowski $\sim$ universal enveloping algebra of $\mathfrak{g}$

- Related group is known to be affine group

$$\mathcal{G} = \mathbb{R} \ltimes_\phi \mathbb{R}^d$$

  – **Not unimodular**: $\exists$ distinct left and right-invariant measures

$$d\nu(s) = \Delta_{\mathcal{G}}(s) d\mu(s), \ \forall s \in \mathcal{G}$$

  modular function (group homomorphism) $\Delta_{\mathcal{G}} : \mathcal{G} \to \mathbb{R}^+_{/0}$
  [see e.g: D. Williams, Math. Surveys and Monographs, Vol. 134, AMS (2007)]

Convolution algebra for κ-Minkowski space

Another quantum space with "Lie algebra-type noncommutativity"

- Lie algebra $\mathfrak{g}$ ($\kappa > 0$): (solvable)

$$[x_0, x_i] = \frac{i}{\kappa} x_i, \quad [x_i, x_j] = 0, \quad i, j = 1, \cdots, d.$$

  Informally, κ-Minkowski $\sim$ universal enveloping algebra of $\mathfrak{g}$

- Related group is known to be affine group

$$\mathcal{G} = \mathbb{R} \ltimes_\phi \mathbb{R}^d$$

  – **Not unimodular**: $\exists$ distinct left and right-invariant measures

$$d\nu(s) = \Delta_{\mathcal{G}}(s) d\mu(s), \ \forall s \in \mathcal{G}$$

  modular function (group homomorphism) $\Delta_{\mathcal{G}} : \mathcal{G} \to \mathbb{R}^+_{/0}$
  [see e.g: D. Williams, Math. Surveys and Monographs, Vol. 134, AMS (2007)]

Convolution algebra for κ-Minkowski space

Right invariant measure. Corresponding convolution and involution

$$(f \circ g)(t) = \int_{\mathcal{G}} d\nu(s)\ f(ts^{-1})g(s),\ \ f^*(t) := \bar{f}(t^{-1})\Delta_{\mathcal{G}}(t),$$

for any functions in $L^1(\mathcal{G})$. Now:
1 – Use group laws for $\mathcal{G}$: $(p, q \in \mathbb{R}^3)$

$$W(p^0, p)W(q^0, q) = W(p^0 + q^0, p + e^{-p^0/\kappa}q)$$

with $\mathbb{I}_{\mathcal{G}} = W(0,0),\ \ W^{-1}(p^0, p) = W(-p^0, -e^{p^0/\kappa}p)$
– Note semi-direct product structure $\mathcal{G} = \mathbb{R} \ltimes_\phi \mathbb{R}^3$, $\phi(p^0)q = e^{-p^0/\kappa}q$
– Functions on $\mathcal{G}$ viewed as functions on $\mathbb{R}^4$ $F(W) = F(p^0, p)$
$F(p^0, p) = \mathcal{F}f(p^0, p) = \int_{\mathbb{R}^4} dx_0 d^3x\ e^{-i(p^0x_0 + p.x)} f(x_0, x)$

Convolution algebra for κ-Minkowski spac

2- Use Modular function

$$\Delta_{\mathcal{G}}(p^0, p) = e^{3p^0/\kappa},$$

– right-invariant measure $d\nu(W) = dp_0 d^3p$ to obtain

$$(f \circ g)(p_0, p) = \int_{\mathbb{R}^4} dq_0 d^3q \; f(p_0 - q_0, p - e^{(q_0 - p_0)/\kappa} q) g(q_0, q)$$

$$f^*(p_0, p) = e^{p_0/\kappa} \bar{f}(-p_0, -e^{p_0/\kappa} p)$$

– Representation of the convolution algebra $\pi(f) = \int_{\mathcal{G}} d\nu(s) f(s) \pi_u(s)$

$$\pi(f \circ g) = \pi(f)\pi(g)$$
$$\pi(f)^\dagger = \pi(f^*)$$

Quantization map and star product

- Quantization map

$$Q(f) := \pi(\mathcal{F}f),$$

- One has

$$\begin{aligned} Q(f \star g) &= Q(f)Q(g) = \pi(\mathcal{F}f)\pi(\mathcal{F}g) = \pi(\mathcal{F}f \circ \mathcal{F}g) \\ Q(f \star g) &= \pi(\mathcal{F}(f \star g)) \end{aligned}$$

yields star product for κ-Minkowski space:

$$f \star g = \mathcal{F}^{-1}(\mathcal{F}f \circ \mathcal{F}g)$$

and involution using $\pi(f)^\dagger = \pi(f^*)$, $Q(f^\dagger) = Q(f)^\dagger$

$$f^\dagger = \mathcal{F}^{-1}(\mathcal{F}(f)^*)$$

Quantization map and star product

- Quantization map

$$Q(f) := \pi(\mathcal{F}f),$$

- One has

$$\begin{aligned} Q(f \star g) &= Q(f)Q(g) = \pi(\mathcal{F}f)\pi(\mathcal{F}g) = \pi(\mathcal{F}f \circ \mathcal{F}g) \\ Q(f \star g) &= \pi(\mathcal{F}(f \star g)) \end{aligned}$$

yields star product for κ-Minkowski space:

$$f \star g = \mathcal{F}^{-1}(\mathcal{F}f \circ \mathcal{F}g)$$

and involution using $\pi(f)^\dagger = \pi(f^*)$, $Q(f^\dagger) = Q(f)^\dagger$

$$f^\dagger = \mathcal{F}^{-1}(\mathcal{F}(f)^*)$$

Star product for 4-d κ-Minkowski space

One obtains ($x := (x_0, \vec{x})$):

$$(f \star g)(x) = \int \frac{dp^0}{2\pi} dy_0 \; e^{-iy_0 p^0} f(x_0 + y_0, \vec{x}) g(x_0, e^{-p^0/\kappa} \vec{x}),$$
$$f^\dagger(x) = \int \frac{dp^0}{2\pi} dy_0 \; e^{-iy_0 p^0} \bar{f}(x_0 + y_0, e^{-p^0/\kappa} \vec{x}),$$

Star product can be extended to multiplier algebra, says $\mathbb{F}(\mathbb{R}^4)$ (as to include in particular coordinates x_μ's, constants,...).
One recovers $[x_0, x_i] = \frac{i}{\kappa} x_i, \quad [x_i, x_j] = 0, \quad i, j = 1, ..., 3$
[Durhuus, Sitarz, J. Noncom.Geom. 7 (2013) 605, Poulain, JCW, PRD (2018) in press arXiv:1801.02715]

Basic properties of the star product

Set $\mathcal{M}_\kappa$ = κ-Minkowski space, $(L^1(\mathcal{G}), \circ) = \mathbb{C}[\mathcal{G}]$

- Star product viewed as (inverse) Fourier transform of convolution product of $\mathbb{C}[\mathcal{G}]$. One has $\mathcal{F} : \mathcal{M}_\kappa \to \mathbb{C}[\mathcal{G}]$
- Star product does not depend on group algebra representation (quantization map does!).
- Funny properties:
If g depends only on x_0, $(f \star g)(x_0, x_1) = f(x_0, x_1)g(x_0)$
If f depends only on x_1, $(f \star g)(x_0, x_1) = f(x_0)g(x_0, x_1)$

Basic properties of the star product

Set $\mathcal{M}_\kappa$ = κ-Minkowski space, $(L^1(\mathcal{G}), \circ) = \mathbb{C}[\mathcal{G}]$

- Star product viewed as (inverse) Fourier transform of convolution product of $\mathbb{C}[\mathcal{G}]$. One has $\mathcal{F} : \mathcal{M}_\kappa \to \mathbb{C}[\mathcal{G}]$
- Star product does not depend on group algebra representation (quantization map does!).
- Funny properties:
  If g depends only on x_0, $(f \star g)(x_0, x_1) = f(x_0, x_1)g(x_0)$
  If f depends only on x_1, $(f \star g)(x_0, x_1) = f(x_0)g(x_0, x_1)$

Basic properties of the star product

Set $\mathcal{M}_\kappa$ = κ-Minkowski space, $(L^1(\mathcal{G}), \circ) = \mathbb{C}[\mathcal{G}]$

- Star product viewed as (inverse) Fourier transform of convolution product of $\mathbb{C}[\mathcal{G}]$. One has $\mathcal{F} : \mathcal{M}_\kappa \to \mathbb{C}[\mathcal{G}]$
- Star product does not depend on group algebra representation (quantization map does!).
- Funny properties:
  If g depends only on x_0, $(f \star g)(x_0, x_1) = f(x_0, x_1)g(x_0)$
  If f depends only on x_1, $(f \star g)(x_0, x_1) = f(x_0)g(x_0, x_1)$

Useful simplying formulas

Simplifying formulas used in the construction of the action functionals:

$$\int d^4x\ (f \star g^\dagger)(x) = \int d^4x\ f(x)\bar{g}(x),\ \int d^4x\ f^\dagger(x) \quad = \int d^4x\ \bar{f}(x),$$

Notice positive maps $\int d^4x : \mathcal{M}_{\kappa+} \to \mathbb{R}^+$ where $\mathcal{M}_{\kappa+}$:

$$\int d^4x\ f \star f^\dagger \geq 0,\ \ \int d^4x\ f^\dagger \star f \geq 0,$$

Convenient Hilbert product on $\mathcal{M}_\kappa$ to construct action functionals

$$\langle f, g\rangle := \int d^4x\ \left(f^\dagger \star g\right)(x) = \int d^4x\ \bar{f}(x)(\sigma \triangleright g)(x)$$

$$\sigma \triangleright f := e^{-\frac{3P_0}{\kappa}} \triangleright f$$

Requirements for the action functional $S_\kappa(\phi, \phi^\dagger)$

Let $\mathcal{P}_\kappa$ be the κ-Poincaré Hopf algebra.
Lukierski, Nowicki, Ruegg, Tolstoy, Phys. Lett. B **268** (1991) 331.

Generated by elements $(P_i, M_i, N_i, \mathcal{E}, \mathcal{E}^{-1})$
$$\mathcal{E} = e^{-P_0/\kappa}$$
Recall $\mathcal{P}_\kappa$ bicrossproduct structure. $\mathcal{P}_\kappa$ has natural action on $\mathcal{M}_\kappa$.
Majid, Ruegg, Phys. Lett. B**334** (1994) 348
$\mathcal{M}_\kappa$ dual of Hopf subalgebra generated by $(P_\mu, \mathcal{E})$
See e.g review Lukierski, J. Phys. Conf. Ser. **804** (2017) 012028

We demand :
– 1 - $S_\kappa(\phi)$ is $\mathcal{P}_\kappa$-invariant, achieved when $S_\kappa(\phi) = \int d^4x\ \mathcal{L}(\phi)$,
– 2 - $S_\kappa(\phi)$ reduces to standard complex ϕ^4 theory in the limit $\kappa \to \infty$.

Trading cyclicity for KMS condition on the field algebra

$\int d^4x$ does not define a trace:

$$\int d^4x\ f \star g = \int d^4x\ (\sigma \triangleright g) \star f$$

$$\sigma \triangleright f := \mathcal{E}^3 \triangleright f = e^{-\frac{3P_0}{\kappa}} \triangleright f$$

– κ-Poincaré invariance implies Twisted trace: cyclicity lost
– Cyclicity is replaced by KMS condition on the algebra of fields = $\mathcal{M}_\kappa$ (not (yet) at the level of algebra of observables).
– The map $f \to \int d^4x\ f(x)$ defines a KMS weight for a particular group of *-automorphims called the modular group.

Definition

Twisted trace on an algebra is a linear positive map Tr such that $\mathrm{Tr}(a \star b) = \mathrm{Tr}((\sigma \triangleright b) \star a)$, where σ is an automorphism of the algebra called the twist.

Representing scalar NCFT as a non-local field theory

– Use simplifying formulas to represent the action as commutative (non-local) field theory.
– Reality condition from $\langle .,.\rangle$ ($\langle f,f\rangle$ is real) selects suitable terms:

$$\langle\phi, K_\kappa\phi\rangle,\quad \langle\phi^\dagger, K_\kappa\phi^\dagger\rangle$$
$$\langle\phi^\dagger\star\phi,\phi^\dagger\star\phi\rangle,\ \langle\phi^\dagger\star\phi^\dagger,\phi^\dagger\star\phi^\dagger\rangle,\ \langle\phi\star\phi^\dagger,\phi\star\phi^\dagger\rangle,\ \langle\phi\star\phi,\phi\star\phi\rangle,$$

$$S_\kappa^{\text{kin}}(\phi^\dagger,\phi) = \langle\phi,(K_\kappa+m^2)\phi\rangle + \langle\phi^\dagger,(K_\kappa+m^2)\phi^\dagger\rangle = \int d^4x\ \phi^\dagger\star(1+\sigma^{-1})(K_\kappa+m^2)\phi,$$

K_κ ΨDO: $(K_\kappa f)(x) = \int \frac{d^4p}{(2\pi)^4} d^4y\ \mathcal{K}_\kappa(p)f(y)e^{ip(x-y)}$

$$S_{1;\kappa}^{\text{int}} = \lambda\int d^4x\left(\phi^\dagger\star\phi\star\phi^\dagger\star\phi\right)(x),\ S_{2;\kappa}^{\text{int}} = \lambda\int d^4x\left(\phi\star\phi\star\phi^\dagger\star\phi^\dagger\right)(x),$$
$$S_{3;\kappa}^{\text{int}} = \lambda\int d^4x\left(\phi\star\phi^\dagger\star\phi\star\phi^\dagger\right)(x),\ S_{4;\kappa}^{\text{int}} = \lambda\int d^4x\left(\phi^\dagger\star\phi^\dagger\star\phi\star\phi\right)(x).$$

Representing scalar NCFT as a non-local field theory

Use explicit expression for star product:

$$S_\kappa[\bar{\phi},\phi] = S_\kappa^{\text{kin}}[\bar{\phi},\phi] + S_{I;\kappa}^{\text{int}}[\bar{\phi},\phi],\ \ I=1,...,4$$

,

$$S_{I;\kappa}^{\text{kin}}[\bar{\phi},\phi] = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\ \bar{\phi}(p)\phi(p)\mathcal{K}(p),$$

$$\mathcal{K}(p) := \frac{1}{2}\left(1+e^{-3p^0/\kappa}\right)\left(K_\kappa(p)+m^2\right),$$

$$S_{I;\kappa}^{\text{int}}[\bar{\phi},\phi] = \lambda\int\left[\prod_{i=1}^4 \frac{d^4p_i}{(2\pi)^4}\right]\bar{\phi}(p_1)\phi(p_2)\bar{\phi}(p_3)\phi(p_4)V_I(p_1,p_2;p_3,p_4)$$

$I=1,...,4$.

Condition 2 OK: $\lim_{\kappa\to\infty} S_\kappa(\phi) =$standard complex ϕ^4 (formally).

Kinetic operators

$(K_\kappa f)(x) = \int \frac{d^4p}{(2\pi)^4} d^4y\ \mathcal{K}_\kappa(p) f(y) e^{ip(x-y)}$ characterized by $\mathcal{K}_\kappa$
– Assumption: Function of the Casimir of $\mathcal{P}_\kappa$, $\mathcal{C}_\kappa(P_\mu)$.
Case 1: $\mathcal{K}_\kappa = \mathcal{C}_\kappa$

$$\mathcal{C}_\kappa(P_\mu) = 4\kappa^2 \sinh^2\left(\frac{P_0}{2\kappa}\right) + e^{P_0/\kappa}\vec{P}^2$$

(Majid-Ruegg basis) $\mathcal{C}_\kappa(P_\mu)$ can be written as (D_0 and D_i self-adjoint):

$$\mathcal{C}_\kappa(P_\mu) = D_0^2 + D_i D^i,\ D_0 := \kappa\mathcal{E}^{-1/2}(1-\mathcal{E}),\quad D_i := \mathcal{E}^{-1/2}P_i,\ \ i = 1,2,3$$

Case 2: $\mathcal{K}_\kappa = K_\kappa^{eq}$ (D_0^{eq}, D_i^{eq} self-adjoints)

$$K_\kappa^{eq}(P_\mu) = D_0^{eq} D_0^{eq} + \sum_i D_i^{eq} D_i^{eq},$$

$$D_0^{eq} := \frac{\mathcal{E}^{-1}}{2}\left(\kappa(1-\mathcal{E}^2) - \frac{1}{\kappa}\vec{P}^2\right)\ ,\ D_i^{eq} := \mathcal{E}^{-1}P_i,$$

Equivariant Dirac operator in D'Andrea, J.Math.Phys. **47** (2006) 062105.

Technical advance: quantum features of NCFT on $\mathcal{M}_\kappa$

- $\rightarrow$ convenient star product for practical use. Well defined procedure using properties of group algebra for $\mathcal{M}_\kappa$.
Easy to represent action as non-local commutative field theory.
- Provides real technical advance opening the actual exploration of NCFT on $\mathcal{M}_\kappa$.
Numerous papers for more than 20 y on classical properties but only very few dealing with quantum properties.
- *Technical* difficulties due to twisted trace can be overcome.

Technical advance: quantum features of NCFT on $\mathcal{M}_\kappa$

- $\rightarrow$ convenient star product for practical use. Well defined procedure using properties of group algebra for $\mathcal{M}_\kappa$.
  Easy to represent action as non-local commutative field theory.
- Provides real technical advance opening the actual exploration of NCFT on $\mathcal{M}_\kappa$.
  Numerous papers for more than 20 y on classical properties but only very few dealing with quantum properties.
- *Technical* difficulties due to twisted trace can be overcome.

Technical advance: quantum features of NCFT on $\mathcal{M}_\kappa$

- $\to$ convenient star product for practical use. Well defined procedure using properties of group algebra for $\mathcal{M}_\kappa$.
  Easy to represent action as non-local commutative field theory.
- Provides real technical advance opening the actual exploration of NCFT on $\mathcal{M}_\kappa$.
  Numerous papers for more than 20 y on classical properties but only very few dealing with quantum properties.
- *Technical* difficulties due to twisted trace can be overcome.

One-loop 2-point functions

[Poulain, JCW, PRD (2018) in press, arXiv:1801.02715]

Systematic examination of 1-loop contributions to 2-point functions of each of the (8) different NCFT:
– Planar, non-planar diagrams still make sense
– Twist effect: controls in part the UV behavior of the contributions. Generates different behaviors among planar and non planar contributions: new sub-type of diagrams.
$\rightarrow$ Summary for K^{eq}: UV divergence milder than commutative ϕ^4.

- $\phi^\dagger \star \phi \star \phi^\dagger \star \phi$
  No UV/IR mixing (no non planar cont.), linear UV divergence
- $\phi \star \phi \star \phi^\dagger \star \phi^\dagger$
  UV/IR mixing signaled by IR singularity, linear UV divergence

Similar conclusions hold for the 2 other NCFT ($\phi \rightarrow \phi^\dagger$).
$\rightarrow$ Casimir: Linear UV divergence$\rightarrow$quadratic

One-loop 2-point functions

[Poulain, JCW, PRD (2018) in press, arXiv:1801.02715]

Systematic examination of 1-loop contributions to 2-point functions of each of the (8) different NCFT:
– Planar, non-planar diagrams still make sense
– Twist effect: controls in part the UV behavior of the contributions. Generates different behaviors among planar and non planar contributions: new sub-type of diagrams.
$\rightarrow$ Summary for K^{eq}: UV divergence milder than commutative ϕ^4.

- $\phi^\dagger \star \phi \star \phi^\dagger \star \phi$
  No UV/IR mixing (no non planar cont.), linear UV divergence
- $\phi \star \phi \star \phi^\dagger \star \phi^\dagger$
  UV/IR mixing signaled by IR singularity, linear UV divergence

Similar conclusions hold for the 2 other NCFT ($\phi \rightarrow \phi^\dagger$).
$\rightarrow$ Casimir: Linear UV divergence$\rightarrow$quadratic

2- and 4-point functions for "orientable" NCFT

[Poulain, JCW, "Beta functions for field theories on κ-Minkowski", to appear]

Interaction $\phi^\dagger \star \phi \star \phi^\dagger \star \phi \ + \ \phi \star \phi^\dagger \star \phi \star \phi^\dagger$ stable against corrections
Kinetic term = K^{eq}.(kinetic term = $\mathcal{C}_\kappa$, nearly similar to ordinary ϕ^4)
Consider first $\phi^\dagger \star \phi \star \phi^\dagger \star \phi$:

- Quadratic part of the effective action:

$$\Gamma^{(2)}[\bar{\phi}, \phi] = \int \frac{d^4p}{(2\pi)^4} \ \bar{\phi}(p)\phi(p) \left(3\omega_1 e^{-3p^0/\kappa} + \omega_2\right)$$

$$\omega_1 \text{ is finite, } \omega_2 = \frac{\lambda\kappa}{2\pi^2} \Lambda_0 + \text{finite terms}$$

- Add $\phi \star \phi^\dagger \star \phi \star \phi^\dagger$. Net effect: $\omega_1 \leftrightharpoons \omega_2$ so that now $\Gamma^{(2)}[\bar{\phi}, \phi] = \int \frac{d^4p}{(2\pi)^4} \ \bar{\phi}(p)\phi(p) \ \omega_{tot} \left(e^{-3p^0/\kappa} + 1\right)$
- 4-point function: Found to be UV finite. Thanks to existence of an estimate for the propagator.

2- and 4-point functions for "orientable" NCFT

[Poulain, JCW, "Beta functions for field theories on κ-Minkowski", to appear]

Interaction $\phi^\dagger \star \phi \star \phi^\dagger \star \phi \ + \ \phi \star \phi^\dagger \star \phi \star \phi^\dagger$ stable against corrections
Kinetic term = K^{eq}.(kinetic term = $\mathcal{C}_\kappa$, nearly similar to ordinary ϕ^4)
Consider first $\phi^\dagger \star \phi \star \phi^\dagger \star \phi$:

- Quadratic part of the effective action:

$$\Gamma^{(2)}[\bar{\phi}, \phi] = \int \frac{d^4p}{(2\pi)^4}\ \bar{\phi}(p)\phi(p) \left(3\omega_1 e^{-3p^0/\kappa} + \omega_2\right)$$

$$\omega_1 \text{ is finite, } \omega_2 = \frac{\lambda\kappa}{2\pi^2}\ \Lambda_0 + \text{finite terms}$$

- Add $\phi \star \phi^\dagger \star \phi \star \phi^\dagger$. Net effect: $\omega_1 \leftrightharpoons \omega_2$ so that now $\Gamma^{(2)}[\bar{\phi}, \phi] = \int \frac{d^4p}{(2\pi)^4}\ \bar{\phi}(p)\phi(p)\ \omega_{tot} \left(e^{-3p^0/\kappa} + 1\right)$
- 4-point function: Found to be UV finite. Thanks to existence of an estimate for the propagator.

2- and 4-point functions for "orientable" NCFT

[Poulain, JCW, "Beta functions for field theories on κ-Minkowski", to appear]

Interaction $\phi^\dagger\star\phi\star\phi^\dagger\star\phi \ + \ \phi\star\phi^\dagger\star\phi\star\phi^\dagger$ stable against corrections
Kinetic term = K^{eq}.(kinetic term = $\mathcal{C}_\kappa$, nearly similar to ordinary ϕ^4)
Consider first $\phi^\dagger\star\phi\star\phi^\dagger\star\phi$:

- Quadratic part of the effective action:

$$\Gamma^{(2)}[\bar{\phi},\phi]=\int\frac{d^4p}{(2\pi)^4}\ \bar{\phi}(p)\phi(p)\left(3\omega_1 e^{-3p^0/\kappa}+\omega_2\right)$$

$$\omega_1 \text{ is finite, } \omega_2=\frac{\lambda\kappa}{2\pi^2}\ \Lambda_0+\text{finite terms}$$

- Add $\phi\star\phi^\dagger\star\phi\star\phi^\dagger$. Net effect: $\omega_1\leftrightharpoons\omega_2$ so that now
$\Gamma^{(2)}[\bar{\phi},\phi]=\int\frac{d^4p}{(2\pi)^4}\ \bar{\phi}(p)\phi(p)\ \omega_{tot}\left(e^{-3p^0/\kappa}+1\right)$
- 4-point function: Found to be UV finite. Thanks to existence of an estimate for the propagator.

Conclusions

- Beta function of orientable model is 0 at 1-loop. Asymptotic safety??
- For 2-point functions: Results agree qualitatively with old results (Grosse, Wohlgenannt, Nucl. Phys. B**748** (2006) 473.)
- Use of present star product very convenient: open the way to investigation of quantum properties of NCFT on $\mathcal{M}_\kappa$.
- Interesting issues/conjectures (among others) to be examined:
  – Orientable model renormalisable to all orders (likely)
  – The fate of UV/IR mixing in other scalar NCFT
  – Extension to gauge theories: twist (twisted trace$\rightarrow$ cyclicity lost) versus gauge invariance of the action.
  – The physical consequences (if any) of twisted trace: implication of KMS condition?

Conclusions

- Beta function of orientable model is 0 at 1-loop. Asymptotic safety??
- For 2-point functions: Results agree qualitatively with old results (Grosse, Wohlgenannt, Nucl. Phys. B**748** (2006) 473.)
- Use of present star product very convenient: open the way to investigation of quantum properties of NCFT on $\mathcal{M}_\kappa$.
- Interesting issues/conjectures (among others) to be examined:
  – Orientable model renormalisable to all orders (likely)
  – The fate of UV/IR mixing in other scalar NCFT
  – Extension to gauge theories: twist (twisted trace$\rightarrow$ cyclicity lost) versus gauge invariance of the action.
  – The physical consequences (if any) of twisted trace: implication of KMS condition?

Conclusions

- Beta function of orientable model is 0 at 1-loop. Asymptotic safety??
- For 2-point functions: Results agree qualitatively with old results (Grosse, Wohlgenannt, Nucl. Phys. B**748** (2006) 473.)
- Use of present star product very convenient: open the way to investigation of quantum properties of NCFT on $\mathcal{M}_\kappa$.
- Interesting issues/conjectures (among others) to be examined:
  – Orientable model renormalisable to all orders (likely)
  – The fate of UV/IR mixing in other scalar NCFT
  – Extension to gauge theories: twist (twisted trace$\rightarrow$ cyclicity lost) versus gauge invariance of the action.
  – The physical consequences (if any) of twisted trace: implication of KMS condition?

Conclusions

- Beta function of orientable model is 0 at 1-loop.
  Asymptotic safety??
- For 2-point functions: Results agree qualitatively with old results (Grosse, Wohlgenannt, Nucl. Phys. B**748** (2006) 473.)
- Use of present star product very convenient: open the way to investigation of quantum properties of NCFT on $\mathcal{M}_\kappa$.
- Interesting issues/conjectures (among others) to be examined:
  – Orientable model renormalisable to all orders (likely)
  – The fate of UV/IR mixing in other scalar NCFT
  – Extension to gauge theories: twist (twisted trace$\rightarrow$ cyclicity lost) versus gauge invariance of the action.
  – The physical consequences (if any) of twisted trace: implication of KMS condition?

Twisted trace and KMS weight [Poulain, JCW, arXiv:1801.02715 (2018)]

– A state is a weight with norm 1. Now a simple lemma:

Lemma (Poulain, JCW - Matassa)

Set $\varphi(f) = \int d^4x\ f(x)$. The map φ defines a KMS weight on $\mathcal{M}_\kappa$ for the modular group of *-automorphisms of $\mathcal{M}_\kappa$

$$\sigma_t(f) := e^{it\frac{3P_0}{\kappa}} \triangleright f = e^{\frac{3t}{\kappa}\partial_0} \triangleright f,$$

– One can check: $\varphi(\sigma_z f) = \varphi(f),\ \varphi(\sigma_{\frac{i}{2}}(f) \star (\sigma_{\frac{i}{2}}(f))^\dagger) = \varphi(f^\dagger \star f)$ where $\sigma_z := e^{iz3P_0/\kappa},\ z \in \mathbb{C}$.

$\sigma = \sigma_{z=i}$ and $\sigma(f^\dagger) = (\sigma^{-1}(f))^\dagger$ (regular automorphism)

Definition (Kustermans)

A KMS weight on a (C*-)algebra $\mathbb{A}$ for a modular group of *-automorphisms $\{\sigma_t\}_{t\in\mathbb{R}}$ is defined as a linear map $\varphi : \mathbb{A}_+ \to \mathbb{R}^+$ such that $\{\sigma_t\}_{t\in\mathbb{R}}$ admits an analytic extension, still a one-parameter group, $\{\sigma_z\}_{z\in\mathbb{C}}$ acting on $\mathbb{A}$ satisfying:

$$\text{i)}\ \ \varphi \circ \sigma_z = \varphi, \quad \text{ii)}\ \ \varphi(a^\dagger \star a) = \varphi(\sigma_{\frac{i}{2}}(a) \star (\sigma_{\frac{i}{2}}(a))^\dagger),$$

Twisted trace and KMS weight [Poulain, JCW, arXiv:1801.02715 (2018)]

– A state is a weight with norm 1. Now a simple lemma:

Lemma (Poulain, JCW - Matassa)

Set $\varphi(f) = \int d^4x\, f(x)$. The map φ defines a KMS weight on $\mathcal{M}_\kappa$ for the modular group of *-automorphisms of $\mathcal{M}_\kappa$

$$\sigma_t(f) := e^{it\frac{3P_0}{\kappa}} \triangleright f = e^{\frac{3t}{\kappa}\partial_0} \triangleright f,$$

– One can check: $\varphi(\sigma_z f) = \varphi(f),\ \varphi(\sigma_{\frac{i}{2}}(f) \star (\sigma_{\frac{i}{2}}(f))^\dagger) = \varphi(f^\dagger \star f)$ where $\sigma_z := e^{iz3P_0/\kappa},\ z \in \mathbb{C}$.

$\sigma = \sigma_{z=i}$ and $\sigma(f^\dagger) = (\sigma^{-1}(f))^\dagger$ (regular automorphism)

Definition (Kustermans)

A KMS weight on a (C*-)algebra $\mathbb{A}$ for a modular group of *-automorphisms $\{\sigma_t\}_{t\in\mathbb{R}}$ is defined as a linear map $\varphi : \mathbb{A}_+ \to \mathbb{R}^+$ such that $\{\sigma_t\}_{t\in\mathbb{R}}$ admits an analytic extension, still a one-parameter group, $\{\sigma_z\}_{z\in\mathbb{C}}$ acting on $\mathbb{A}$ satisfying:

$$\text{i)}\ \ \varphi \circ \sigma_z = \varphi, \quad \text{ii)}\ \ \varphi(a^\dagger \star a) = \varphi(\sigma_{\frac{i}{2}}(a) \star (\sigma_{\frac{i}{2}}(a))^\dagger),$$

KMS condition [Poulain, JCW, arXiv:1801.02715 (2018)]

– (up to technical conditions): If $\varphi(f)$ is a KMS weight, one has:

$$f_{a,b}(t) = \int d^4x \; \sigma_t(a) \star b := \langle \sigma_t(a) \star b \rangle = \langle b \star \sigma_{t-i}(a) \rangle$$

– looks like usual KMS condition among correlation functions.
– But here, only the algebra of fields (not the one of the observables).
– For quantum systems, $f_{A,B}(t) = \langle \Sigma_t(A) B \rangle_\Omega$, (thermal) vacuum Ω, A, B functionals of fields, Σ_t evolution operator, in observables algebra.
– If a KMS condition holds for algebra of observables, flow generated by the modular group σ_t may be used to define a global time.
– Tomita-Takesaki modular theory. $\Delta_T = e^{3P_0/\kappa}$ Tomita operator
– "physical" KMS condition on observables coming from above KMS?

Presently under study

2-d Moyal star product from a group algebra

- 1) Convolution algebra of Heisenberg group : $(L^1(\mathbb{H}), \circ) := \mathbb{C}[\mathbb{H}]$
  Lie algebra: 3-d Heisenberg Lie algebra $[P, Q] = iZ$, Z central.
  - Group laws for $\mathbb{H}$:
    $g(z, u, v)g(z', u', v') = h(z + z' + \frac{1}{2}(uv' - u'v), u + u', v + v')$,
    $g^{-1}(z, u, v) = g(-z, -u, -v)$,
  - $\mathbb{H}$ unimodular. Haar measure: $d\mu(h) = dzdqdp$ (Lebesgues).
  - Convolution product $(f \circ g)(t) = \int_{\mathbb{H}} d\mu(s) f(s) g(s^{-1}t)$.
  - At this stage, functions on $\mathbb{H}$ viewed as functions on $\mathbb{R}^3$.
- 2) Representation of convolution algebra $\pi : \mathbb{C}[\mathbb{H}] \to \mathcal{B}(L^2(\mathbb{R}))$

$$\pi(f) = \int_{\mathbb{R}^3} dzdudv\ f(z, u, v)\pi_\hbar[g(z, u, v)]$$

$$(\pi_\hbar[g(z, u, v)]\psi)(x) = e^{i\frac{\hbar}{2}z}.e^{i(\hbar\frac{uv}{2}+vx)}\psi(x + \hbar u),$$

unirreps of $\mathbb{H}$: ($\hbar \neq 0$ (Stone-von Neumann))

$$\pi(f \circ g) = \pi(f)\pi(g)$$

2-d Moyal star product from a group algebra

- 1) Convolution algebra of Heisenberg group : $(L^1(\mathbb{H}), \circ) := \mathbb{C}[\mathbb{H}]$
  Lie algebra: 3-d Heisenberg Lie algebra $[P, Q] = iZ$, Z central.
  - Group laws for $\mathbb{H}$:
    $g(z, u, v)g(z', u', v') = h(z + z' + \frac{1}{2}(uv' - u'v), u + u', v + v')$,
    $g^{-1}(z, u, v) = g(-z, -u, -v)$,
  - $\mathbb{H}$ unimodular. Haar measure: $d\mu(h) = dzdqdp$ (Lebesgues).
  - Convolution product $(f \circ g)(t) = \int_{\mathbb{H}} d\mu(s) f(s) g(s^{-1}t)$.
  - At this stage, functions on $\mathbb{H}$ viewed as functions on $\mathbb{R}^3$.
- 2) Representation of convolution algebra $\pi : \mathbb{C}[\mathbb{H}] \to \mathcal{B}(L^2(\mathbb{R}))$

$$\pi(f) = \int_{\mathbb{R}^3} dzdudv\ f(z, u, v)\pi_h[g(z, u, v)]$$

$$(\pi_h[g(z, u, v)]\psi)(x) = e^{i\frac{\hbar}{2}z}.e^{i(\hbar\frac{uv}{2}+vx)}\psi(x + \hbar u),$$

unirreps of $\mathbb{H}$: ($\hbar \neq 0$ (Stone-von Neumann))

$$\pi(f \circ g) = \pi(f)\pi(g)$$

2-d Moyal star product from a group algebra

- 1) Convolution algebra of Heisenberg group : $(L^1(\mathbb{H}), \circ) := \mathbb{C}[\mathbb{H}]$
  Lie algebra: 3-d Heisenberg Lie algebra $[P, Q] = iZ$, Z central.
  - Group laws for $\mathbb{H}$:
    $g(z, u, v)g(z', u', v') = h(z + z' + \frac{1}{2}(uv' - u'v), u + u', v + v')$,
    $g^{-1}(z, u, v) = g(-z, -u, -v)$,
  - $\mathbb{H}$ unimodular. Haar measure: $d\mu(h) = dzdqdp$ (Lebesgues).
  - Convolution product $(f \circ g)(t) = \int_{\mathbb{H}} d\mu(s) f(s) g(s^{-1}t)$.
  - At this stage, functions on $\mathbb{H}$ viewed as functions on $\mathbb{R}^3$.
- 2) Representation of convolution algebra $\pi : \mathbb{C}[\mathbb{H}] \to \mathcal{B}(L^2(\mathbb{R}))$

$$\pi(f) = \int_{\mathbb{R}^3} dzdudv\ f(z, u, v)\pi_h[g(z, u, v)]$$

$$(\pi_h[g(z, u, v)]\psi)(x) = e^{i\frac{\hbar}{2}z}.e^{i(\hbar\frac{uv}{2}+vx)}\psi(x + \hbar u),$$

unirreps of $\mathbb{H}$: ($\hbar \neq 0$ (Stone-von Neumann))

$$\pi(f \circ g) = \pi(f)\pi(g)$$

2-d Moyal star product from a group algebra

- 1) Convolution algebra of Heisenberg group : $(L^1(\mathbb{H}),\circ) := \mathbb{C}[\mathbb{H}]$
  Lie algebra: 3-d Heisenberg Lie algebra $[P,Q]=iZ$, Z central.
  - Group laws for $\mathbb{H}$:
    $g(z,u,v)g(z',u',v') = h(z+z'+\frac{1}{2}(uv'-u'v),u+u',v+v')$,
    $g^{-1}(z,u,v)=g(-z,-u,-v)$,
  - $\mathbb{H}$ unimodular. Haar measure: $d\mu(h)=dzdqdp$ (Lebesgues).
  - Convolution product $(f\circ g)(t)=\int_{\mathbb{H}} d\mu(s)f(s)g(s^{-1}t)$.
  - At this stage, functions on $\mathbb{H}$ viewed as functions on $\mathbb{R}^3$.
- 2) Representation of convolution algebra $\pi:\mathbb{C}[\mathbb{H}]\to\mathcal{B}(L^2(\mathbb{R}))$

$$\pi(f)=\int_{\mathbb{R}^3} dzdudv\ f(z,u,v)\pi_\hbar[g(z,u,v)]$$

$$(\pi_\hbar[g(z,u,v)]\psi)(x)=e^{i\frac{\hbar}{2}z}.e^{i(\hbar\frac{uv}{2}+vx)}\psi(x+\hbar u),$$

unirreps of $\mathbb{H}$: ($\hbar\neq 0$ (Stone-von Neumann))

$$\pi(f\circ g)=\pi(f)\pi(g)$$

2-d Moyal star product from a group algebra

- 1) Convolution algebra of Heisenberg group : $(L^1(\mathbb{H}),\circ) := \mathbb{C}[\mathbb{H}]$
Lie algebra: 3-d Heisenberg Lie algebra $[P,Q]=iZ$, Z central.
    - Group laws for $\mathbb{H}$:
    $g(z,u,v)g(z',u',v')=h(z+z'+\frac{1}{2}(uv'-u'v),u+u',v+v')$,
    $g^{-1}(z,u,v)=g(-z,-u,-v)$,
    - $\mathbb{H}$ unimodular. Haar measure: $d\mu(h)=dzdqdp$ (Lebesgues).
    - Convolution product $(f\circ g)(t)=\int_{\mathbb{H}}d\mu(s)f(s)g(s^{-1}t)$.
    - At this stage, functions on $\mathbb{H}$ viewed as functions on $\mathbb{R}^3$.
- 2) Representation of convolution algebra $\pi:\mathbb{C}[\mathbb{H}]\to\mathcal{B}(L^2(\mathbb{R}))$

$$\pi(f)=\int_{\mathbb{R}^3}dzdudv\ f(z,u,v)\pi_\hbar[g(z,u,v)]$$

$$(\pi_\hbar[g(z,u,v)]\psi)(x)=e^{i\frac{\hbar}{2}z}.e^{i(\hbar\frac{uv}{2}+vx)}\psi(x+\hbar u),$$

unirreps of $\mathbb{H}$: ($\hbar\neq 0$ (Stone-von Neumann))

$$\pi(f\circ g)=\pi(f)\pi(g)$$

2-d Moyal star product from a group algebra

- 1) Convolution algebra of Heisenberg group : $(L^1(\mathbb{H}), \circ) := \mathbb{C}[\mathbb{H}]$
Lie algebra: 3-d Heisenberg Lie algebra $[P, Q] = iZ$, Z central.
    - Group laws for $\mathbb{H}$:
    $g(z, u, v)g(z', u', v') = h(z + z' + \frac{1}{2}(uv' - u'v), u + u', v + v')$,
    $g^{-1}(z, u, v) = g(-z, -u, -v)$,
    - $\mathbb{H}$ unimodular. Haar measure: $d\mu(h) = dzdqdp$ (Lebesgues).
    - Convolution product $(f \circ g)(t) = \int_{\mathbb{H}} d\mu(s)f(s)g(s^{-1}t)$.
    - At this stage, functions on $\mathbb{H}$ viewed as functions on $\mathbb{R}^3$.
- 2) Representation of convolution algebra $\pi : \mathbb{C}[\mathbb{H}] \to \mathcal{B}(L^2(\mathbb{R}))$

$$\pi(f) = \int_{\mathbb{R}^3} dzdudv\ f(z, u, v)\pi_h[g(z, u, v)]$$

$$(\pi_h[g(z, u, v)]\psi)(x) = e^{i\frac{\hbar}{2}z}.e^{i(\hbar\frac{uv}{2}+vx)}\psi(x + \hbar u),$$

unirreps of $\mathbb{H}$: ($\hbar \neq 0$ (Stone-von Neumann))

$$\pi(f \circ g) = \pi(f)\pi(g)$$

2-d Moyal star product from a group algebra

Setting:

$$f^\#(u,v) := \int dz\ f(z,u,v)e^{i\frac{\hbar}{2}z} \text{ defines a map } \# : L^1(\mathbb{R}^3) \to L^1(\mathbb{R}^2)$$

and $(\pi(f)\psi)(x) = \int_{\mathbb{R}^2} dudv\ f^\#(u,v)e^{i(bx+\hbar\frac{pq}{2})}\psi(x+\hbar u)$.
Consider action of $\#$ on convolution product. Standard calculation :

$$(f\circ g)^\#(u,v) = \int_{\mathbb{R}^2} du'dv'\ f^\#(u',v')g^\#(u-u',v-v')e^{\frac{i}{2}(uv'-u'v)}$$

yields twisted convolution: $(f\circ g)^\#(u,v) = (f^\#\hat{\circ}g^\#)(u,v)$.
$f^\#(u,v) \sim$ functions on momentum space. Set $f^\#(u,v) = \mathcal{F}f(u,v)$

- 3) Weyl quantization map $W(f) := \pi(\mathcal{F}f)$, $W(f\star g) = W(f)W(g)$ yields

$$f\star g = \mathcal{F}^{-1}(\mathcal{F}f\hat{\circ}\mathcal{F}g)$$

whose expression is the usual Moyal product.