Nonassociative deformation quantization

Dmitri Vassilevich

UFABC

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Based on works with

Vlad Kupriyanov and Fernando Martins Costa Oliveira

Support: CNPq, FAPESP

Bayen, Flato, Fronsdal, Lichenrovicz and Sternheimer (1977):
Data: a manifold M with a bi-vector P defining a Poisson bracket $\{f,g\} = P(df, dg)$ for $f, g \in C^\infty(M)$.

Star product: a multiplication on $C^\infty(M)[[\lambda]]$:

$$f \star g = f \cdot g + \sum_r \lambda^r C_r(f,g)$$

where C_r's are bidifferential operators, and

$$C_1(f,g) = \{f,g\}$$

Normally, this product is assumed to be associative. This implies that $\{\ ,\ \}$ is a Poisson-Lie bracket.

Developments

Fedosov [around 1990]: Weyl bundles over symplectic manifolds and quantization of an arbitrary symplectic Poisson bracket.

Kontsevich [1997]: a one-to-one correspondence between equivalence classes of Poisson-Lie structures and equivalence classes of the star products (L_∞ quasi isomorphism).

Schomerus, Seiberg, Witten ....[1999]: Correlation functions of open strings are given by star products.

However, [around 2001]: the corresponding Poisson structures are not necessarily Poisson-Lie.

New reincarnation [Munich group, 2011] : strings on so called "non-geometric backgrounds" require Poisson structures that are never Poisson-Lie.

A good context for dealing with non-Lie-Poisson structures is quantization of gerbes and algebroid stacks [Ševera, Aschieri, Schupp, Bresler, Nest, Tsygan]. Caveat: there is no obvious place for these structures in string theory.

Therefore, we are back at the problem of non-associative deformation quantization of which very little is known in general, though there are many examples [ask Richard Szabo].

Weyl products

Let $M = \mathbb{R}^n$, and let x^j be coordinates on $\mathbb{R}^n$. Quantization replaces x^j with noncommuting objects $\hat{x}^j$ (quantum mechanical operators). This correspondence may be extended to smooth functions by means of the Weyl symmetric ordering

$$f \to \hat{f} = W(f(x))$$

E.g., $W(x^i x^j) = \frac{1}{2}(\hat{x}^i \hat{x}^j + \hat{x}^j \hat{x}^i)$. Then, the Weyl star product of functions is defined through the composition of corresponding operators:

$$W(f \star_W g) = W(f) \circ W(g).$$

Obviously, only associative star products may be obtained in this way.

Nonassociative case [Kupriyanov & D.V, 2015]
Obs.: any start product defines a formal differential operator $\hat{f}$ for any function f through the relation

$$\hat{f} \cdot g := f \star g$$

We request that

$$\hat{f} = W(f(x))$$

for all f. (For the associative case, this is equivalent to $\star$ being a Weyl star product. However, this does not assume associativity).
We expand in the derivatives

$$\hat{x} = x + \sum_{k=1}^{\infty} \Gamma^{(k)}(\lambda, x)(\lambda\partial)^{(k)}$$

We also assume:

1. stability of the unity $1 \star f = f \star 1 = f$ (which is equivalent to certain symmetry properties of $\Gamma^{(k)}$;
2. weak Hermiticity $(x \star f)^* = f^* \star x$ (symmetry assumption on C_r),
3. strict triangularity (restriction on the order of differential operators appearing in Γ and a requirement that the leading term with two derivatives in $\star$ product has no corrections at higher orders of λ).

Then:

Theorem Under these assumptions, there is unique Weyl star product for any smooth Poisson bracket on $\mathbb{R}^n$.

Proof: Explicit iterative construction.

Remarks:

- This star product indeed describes some correlation functions of open strings.
- The construction does not tell us anything on what kind of an algebra we got. E.g., it is not clear whether it satisfies any polynomial identity.

Nearly associative star products

(F.M.C.Oliveira and D.V., arXiv:1802.05808, to appear in LMP)

The associator:

$$A(f,g,h) \equiv f \star (g \star h) - (f \star g) \star h$$

An algebra is

- alternative, if A is totally anti-symmetric
- right alternative, if $A(f,g,g) = 0$
- flexible, if $A(f,g,f) = 0$

These algebras, together with Jordan algebras, are called nearly associative.

Right alternative algebras:

Lemma: Deformation quantization algebras have no nilpotent elements.

Theorem (Kleinfeld) Every right alternative algebra without nilpotent elements is alternative.

(This kills right alternative algebras as an independent possibility)

Alternative algebras

What is (not) known:

- There are no examples of alternative start products.
- [Bojowald et al, 2016] There are no alternative monopole star products (monopole=a bunch of assumptions)

Besides: [Kupriyanov, 2016] For an alternative star product, the jacobiator

$$\{f, g, h\} := \{f, \{g, h\}\} + \text{cyclic permutations}$$

has to satisfy the Malcev identity $\{h, f, \{h, g\}\} = \{\{h, f, g\}, h\}$ (making $\{.,.\}$ a Poisson-Malcev structure).

Theorem [Shestakov, 2000] Poisson-Malcev algebras satisfy the identity: $\{f, g, h\} \cdot \{f, g\} = 0$.

Applying this to Poisson brackets om $\mathbb{R}^n$, one obtains that $\{f, g, h\} \cdot \{f, g\} = 0$, i.e. alternative star products cannot provide a deformation quantization on non-Lie Poisson structures. (Which makes them much less interesting in the string theory context).

In all alternative algebras there is the following identity:

$$(A([f,g]^2,h,r))^2 = 0.$$

Then we use the Weinstein splitting theorem for Poisson-Lie structures to show that $[f,g]^2$ is sufficiently generic if the Poisson bivector $P^{ij} \neq 0$ and to get that $A = 0$. Consequently, an alternative star product is associative if at least one of the following assumptions holds

- $P \neq 0$ on a dense set of $\mathbb{R}^n$
- the symbols of bidifferential operators C_r are local polynomials of P.

For all practical purposes this means that alternative star products are associative!

Flexible algebras

- **Anderson, 1966**: An algebra is flexible if and only if the commutator $[h,.]$ is a derivation on the algebra with the same linear space and with the Jordan product $f \circ g \equiv f \star g + g \star f$. Good news for defining Schrödinger-like evolution equations.
- For any Poisson bracket there is always a (quite trivial) star product

$$f \star g = f \cdot g + \lambda \{f, g\}$$

  No other examples are known.

Can an associative star product be deformed to a flexible one?