Noncommutative Gauge Theories on D-Branes in Non-Geometric Backgrounds

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Quantum Structure of Spacetime

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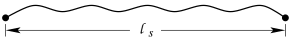
Outline

- Introduction/Motivation
- Review: D-branes in constant B-fields
- Non-geometric backgrounds:
  Expectations from topological T-duality
- Twisted tori & D-branes in T-folds
- Doubled twisted tori
  & D-branes in locally non-geometric backgrounds

Work in progress with Chris Hull

String Geometry



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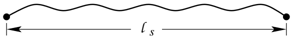

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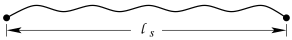

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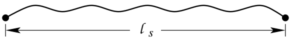

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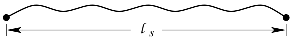

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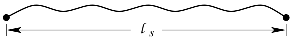

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- Not all spacetime geometries are ordinary geometric spaces, e.g. noncommutative spaces can arise as decoupling limits
- Use effective field theories as probes of geometry: Introduce D-branes and take decoupling limit $\Longrightarrow$ Noncommutative worldvolume gauge theories in an NS–NS B-field background

  [See Erik Plauschinn's talk for a direct open string perspective]

Open String Dynamics in Constant B-Fields

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- 2-point function on boundary of disk: ordering

$$\left\langle x^i(t)\, x^j(t')\right\rangle = -\alpha'\, G^{ij} \log(t-t')^2 + \tfrac{\mathrm{i}}{2}\, \theta^{ij}\, \mathrm{sgn}(t-t')$$

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- In decoupling limit $\alpha' \sim \epsilon^{1/2}$, $g_{ij} \sim \epsilon$ with $\epsilon \longrightarrow 0$:

$$G = -(2\pi\,\alpha')^2\, B\, g^{-1}\, B \qquad , \qquad \theta = B^{-1}$$

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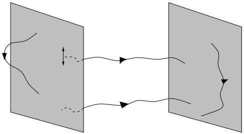

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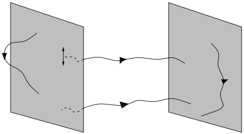

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- Effective Yang-Mills coupling in Dp-brane gauge theory:

$$g_{\mathrm{YM}}^2 = \frac{(2\pi)^{p-2}}{(\alpha')^{(3-p)/2}}\; g_s\, \mathrm{e}^{\,\phi} \left(\frac{\det(g + 2\pi\,\alpha'\,B)}{\det g}\right)^{1/2}$$

Finite in decoupling limit if $g_s\,\mathrm{e}^{\,\phi} \sim \epsilon^{(3-p+r)/4}$, $\;r = \mathrm{rank}(B)$

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  $$\mathcal{E}' = (A\,\mathcal{E} + B)\,\frac{1}{C\,\mathcal{E} + D} \quad \text{for} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SO(p,p;\mathbb{Z})$$

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$$G' = (C\,\theta + D)\,G\,(C\,\theta + D)^\top \quad , \quad \theta' = (A\,\theta + B)\,\frac{1}{C\,\theta + D}$$

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- Refinement of topological T-duality via Morita equivalence of noncommutative tori: $K(T^p_\theta) = K(T^p_{\theta'})$

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- **Goal:** Understand worldvolume gauge theories in these non-geometric backgrounds [extending (Lowe, Natase & Ramgoolam '03; Ellwood & Hashimoto '06; Grange & Schäfer-Nameki '07)]; compare with noncommutative/nonassociative closed string geometry (Blumenhagen & Plauschinn '10; Lüst '10; Blumenhagen, Deser, Lüst, Plauschinn & Rennecke '11; Mylonas, Schupp & RS '12; . . . ; cf. Laurent Freidel's talk)

Expectations from Topological T-Duality

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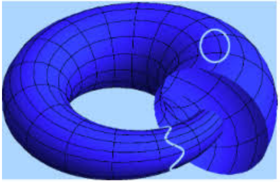

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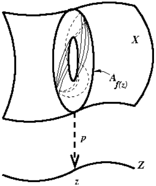

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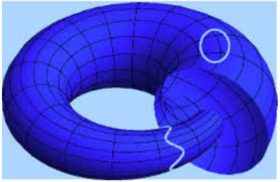

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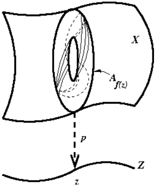

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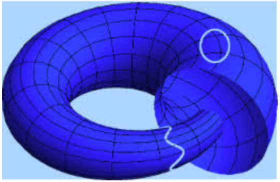

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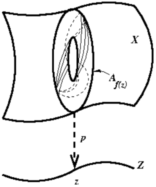

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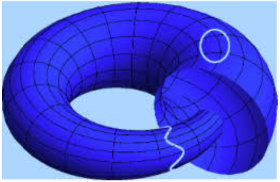

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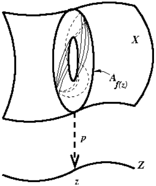

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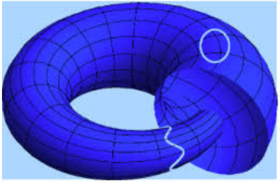

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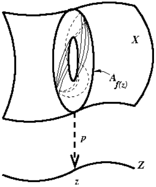

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- Q-flux $(d = 2)$: $\widehat{\mathcal{A}} = C^*(G_{\mathbb{Z}}) \otimes \mathcal{K} =$ field of noncommutative 2-tori $T^2_{\theta(x)}$, $\theta(x) = m\,x$, $x \in S^1$ T-fold

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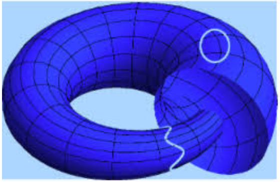

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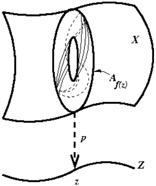

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- R-flux $(d=3)$: $\widehat{\mathcal{A}} = \mathcal{K}\big(L^2(\widehat{T}^3)\big) \rtimes_{u_\phi} \widehat{T}^3 =$ nonassociative 3-torus T^3_ϕ , $\phi \in Z^3\big(\widehat{T}^3, U(1)\big)$ associated to H locally non-geometric

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$$\Gamma \subseteq O(2,2;\mathbb{Z}) \simeq \big(SL(2,\mathbb{Z})_\tau \times SL(2,\mathbb{Z})_\rho\big) \rtimes \mathbb{Z}_2$$

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- For $\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$: $\tau(x+1) = \mathcal{M}\big[\tau(x)\big] = \dfrac{a\,\tau(x)+b}{c\,\tau(x)+d}$

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- Conjugacy classes of $SL(2, \mathbb{Z})$:
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  2. Elliptic: $\mathrm{Tr}(\mathcal{M}) < 2$
  3. Hyperbolic: $\mathrm{Tr}(\mathcal{M}) > 2$

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g &= (2\pi\, r)^2\, \mathrm{d}x^2 + \frac{\tau_2(x)}{|\tau(x)|^2} \Big(A \left(\mathrm{d}y^1\right)^2 + \frac{(2\pi\, \alpha')^2}{A} \left(\mathrm{d}y^2\right)^2 \Big) \\
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- Apply open-closed string transformation:

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\theta &= \tau_1(x)\, \partial_{y^1} \wedge \partial_{y^2}
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- There is a consistent decoupling limit of the D2-brane on the T-fold with $\alpha', A, \tau_2^\circ, g_s \sim \epsilon^{1/2}$ such that as $\epsilon \longrightarrow 0$:

$$G = \big(2\pi\,r_1\,\mathrm{d}y^1\big)^2 + \big(2\pi\,r_2\,\mathrm{d}y^2\big)^2$$

$$\theta(x) = m\,x \qquad , \qquad g^2_{\rm YM} = 2\pi\,\bar{g}_s$$

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- Open strings see conventional geometric T^3 with non-geometric noncommutativity $\theta(x)$!
  (cf. Morita equivalence symmetry of noncommutative Yang-Mills theory is inherited from T-duality in decoupling limit)

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$$\theta(x) = \tan\left(\tfrac{m\,\pi}{2}\,x\right) \qquad , \qquad g_{\rm YM}(x)^2 = 2\pi\,\bar{g}_s\,\Big|\cos\left(\tfrac{m\,\pi}{2}\,x\right)\Big|$$

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- Orbifold point in moduli space: $\tau(x)=\mathrm{i}$ for all $x\in S^1$ when $\tau^\circ=\mathrm{i}$; Open and closed string modes cannot be decoupled, worldvolume gauge theory is ordinary Yang-Mills theory on a conventional geometric torus

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- T-duality along ∂_{y^2} plus Fourier transform $\implies w^2 \longmapsto p_2 \longmapsto y^2$ $\implies f \star \tilde{f}$ in D2-brane gauge theory in T-fold

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- T-duality realized linearly by $O(3,3;\mathbb{Z})$-transformations, use to follow orbits of D-branes in doubled torus geometry (Lawrence, Schulz & Wecht '06; Albertsson, Kimura & Reid-Edwards '08)

R-Flux and Noncommutative Gauge Theory

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Background	Dp-brane	x	y^1	y^2	$\widetilde{x}$	$\widetilde{y}_1$	$\widetilde{y}_2$
H-flux	D0-brane	$-$	$-$	$-$	$\times$	$\times$	$\times$
f-flux	D1-brane	$-$	$\times$	$-$	$\times$	$-$	$\times$
Q-flux	D2-brane	$-$	$\times$	$\times$	$\times$	$-$	$-$
R-flux	D3-brane	$\times$	$\times$	$\times$	$-$	$-$	$-$

R-Flux and Noncommutative Gauge Theory

Background	Dp-brane	x	y^1	y^2	$\widetilde{x}$	$\widetilde{y}_1$	$\widetilde{y}_2$
H-flux	D0-brane	—	—	—	×	×	×
f-flux	D1-brane	—	×	—	×	—	×
Q-flux	D2-brane	—	×	×	×	—	—
R-flux	D3-brane	×	×	×	—	—	—

- Decoupling limit in R-space additionally requires $r \sim \epsilon^{1/2}$, with open string noncommutative geometry:

$$
\begin{aligned}
G_R &= (2\pi\, \bar{r}_x\, \mathrm{d}x)^2 + G_{\mathrm{D2}}\big|_{x \to \widetilde{x}} \\
\theta_R &= \tau_1(\widetilde{x})\big|_{\tau_2^\circ = 0}\ \partial_{y^1} \wedge \partial_{y^2}
\end{aligned}
$$

R-Flux and Noncommutative Gauge Theory

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f-flux	D1-brane	$-$	$\times$	$-$	$\times$	$-$	$\times$
Q-flux	D2-brane	$-$	$\times$	$\times$	$\times$	$-$	$-$
R-flux	D3-brane	$\times$	$\times$	$\times$	$-$	$-$	$-$

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- Noncommutative D3-brane gauge theory in $\mathscr{X}$ returns to itself under $\widetilde{x}\longmapsto\widetilde{x}+1$ up to Morita equivalence