Thermodynamic relations for black holes with NLE fields

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- I. Barjašić, L. Gulin and I.S.: Nonlinear electromagnetic fields and symmetries
  PRD **95** (2017) 124037 [1705.00628]

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- L. Gulin and I.S.: Generalizations of the Smarr formula for black holes with nonlinear electromagnetic fields
  CQG **35** (2018) 025015 [**1710.04660**]

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- L. Gulin and I.S.: Generalizations of the Smarr formula for black holes with nonlinear electromagnetic fields
  CQG **35** (2018) 025015 [**1710.04660**]

- I.S.: Spacetimes dressed with stealth electromagnetic fields
  PRD **97** (2018) 084041 [**1711.07490**]

Central question:

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How does **nonlinear electrodynamics** modify black hole thermodynamics?

Black hole thermodynamics

	THERMODYNAMICS	BLACK HOLES
0.	$T = \text{const.}$	$\kappa = \text{const.}$
1.	$\mathrm{d}E = T\mathrm{d}S + \dots$	$\mathrm{d}M = \frac{1}{8\pi}\,\kappa\,\mathrm{d}A + \dots$
2.	$\delta S \geq 0$	$\delta A \geq 0$
3.	$T \nrightarrow 0$	$\kappa \nrightarrow 0$
Euler	$E = TS + \dots$	$M = \frac{1}{4\pi}\,\kappa\,A + \dots$

Going beyond Einstein and Maxwell

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- develop algorithms for the extensions of thermodynamic relations

NLE redux

Basic assumptions

- (mostly) natural system of units: $c = G = 4\pi\varepsilon_0 = 4\pi\mu_0 = 1$

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$$F \wedge *F = \frac{1}{2}\,\mathcal{F} * 1 \qquad\qquad F \wedge F = \frac{1}{2}\,\mathcal{G} * 1$$

Maxwell's electrodynamics

- Maxwell's Lagrangian

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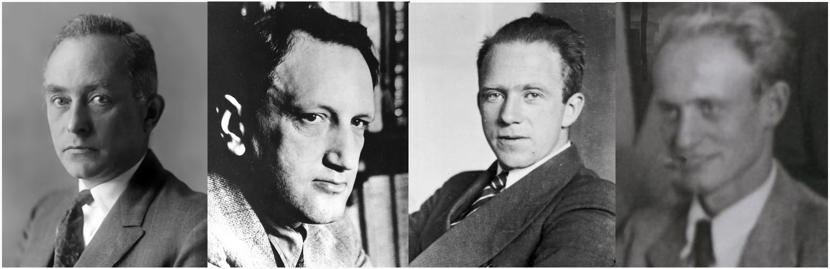
- energy-momentum tensor

$$T^{(\mathsf{Max})}_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}\,g_{ab}\mathcal{F}\right)$$

Inception of NLE in 1930s



Max Born, Leopold Infeld, Werner K. Heisenberg, Hans H. Euler

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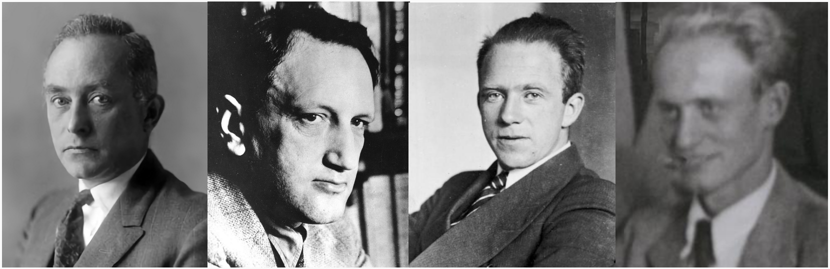

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- Max Born (1933), introducing an upper limit b for the field strength

$$\mathscr{L}^{(\mathrm{Born})} = b^2 \left(1 - \sqrt{1 + \frac{\mathcal{F}}{2b^2}} \right)$$

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- Born-Infeld (1934)

$$\mathscr{L}^{(\mathrm{BI})} = b^2 \left(1 - \sqrt{1 + \frac{\mathcal{F}}{2b^2} - \frac{\mathcal{G}^2}{16b^4}} \right)$$

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- alternative route ("reverse engineering")

$$E = \frac{Q}{r^2}\,\mathrm{d}r \quad \to \quad E = \frac{Q}{\sqrt{r^4 + (Q/b)^2}}\,\mathrm{d}r \quad \to \quad \mathscr{L} = \dots$$

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- Euler-Heisenberg (1935): one-loop QED corrections to Maxwell

$$\mathscr{L}^{(\mathsf{EH})} = -\frac{1}{4}\,\mathcal{F} + \frac{\alpha^2}{360 m_e^4}\left(4\mathcal{F}^2 + 7\mathcal{G}^2\right) + O(\alpha^3)$$

Proliferation of NLE models

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- Bardeen's model (1968)

$$\mathscr{L} = \frac{3M}{g^3}\left(\frac{g\sqrt{2\mathfrak{F}}}{2+g\sqrt{2\mathfrak{F}}}\right)^{5/2}$$

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"Bardeen's black hole": $F = g\,\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi$

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\,\mathrm{d}\Omega^2\ , \quad f(r) = 1 - \frac{2Mr^2}{(r^2+g^2)^{3/2}}$$

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- NLE zoo:

  [Soleng: PRD **52** (1995)] $\mathscr{L} \sim \ln(1+\lambda\mathcal{F})$

  [Hassaine, Martínez: PRD **75** (2007)] $\mathscr{L} \sim \mathcal{F}^s$

  [Hendi: JHEP **03** (2012)] $\mathscr{L} \sim \exp(-\mathcal{F}/\beta^2) - 1$

Experimental constraints on NLE

$$E_{\mathsf{lightning}} \sim 10^6\,\mathrm{V/m}\ , \quad E_{\mathsf{cr}}^{(\mathsf{EH})} = \frac{m_e^2 c^3}{e\hbar} \sim 10^{18}\,\mathrm{V/m}$$

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  remaining window $10^{19}\,\mathrm{V/m} \lesssim b \lesssim 10^{27}\,\mathrm{V/m}$ [EPJC **78** (2018)]

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  [Rev.Mod.Phys. **82** (2010)]
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- Fouché, Battesti and Rizzo: *Limits on nonlinear electrodynamics* [PRD **93** (2016)]

Basic elements of NLE

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- energy-momentum tensor

$$T_{ab} = -4\mathscr{L}_\mathcal{F}\, T_{ab}^{(\mathrm{Max})} + \frac{1}{4}\, T g_{ab}$$

Zeroth law of BH thermodynamics

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  proofs:

  via Frobenius theorem (Carter 1972)

  via DEC and Einstein EOM (Bardeen et al. 1973)

  via bifurcation surface (Kay & Wald 1991)

Electromagnetic scalar potentials

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- Electric and magnetic 1-forms with respect to K^a

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- **global** condition, $H^1_{\mathrm{dR}}(\langle\langle M \rangle\rangle) = 0$, guarantees

$$E = -\mathrm{d}\Phi \qquad\qquad H = -\mathrm{d}\Psi$$

Symmetry inheritance

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Basic trick (for minimally coupled fields)

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$$
\begin{array}{ccc}
\pounds_{\mathrm{K}} E_{ab} = 0 & \xrightarrow{\color{darkred}{E_{ab} = 8\pi T_{ab}}} & \pounds_{\mathrm{K}} T_{ab} = 0 \\
\uparrow & & \downarrow \\
\pounds_{\mathrm{K}} g_{ab} = 0 & & \pounds_{\mathrm{K}} \psi^{a\ldots}{}_{b\ldots} = \cdots
\end{array}
$$

Symmetry inheritance of the EM field in a nutshell

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(1+1)	$\pounds_{\mathrm{K}} F_{ab} = 0$	BGS '17
(1+2)	$\pounds_{\mathrm{K}} F_{ab} = 0$	CDPS '16
(1+3)	$\pounds_{\mathrm{K}} F_{ab} = \alpha \, {*F}_{ab}$	MW '75 / WY '76
≥ 5	$\pounds_{\mathrm{K}} (F_{ac} {F_b}^c) = 0$	BGS '17

[PRD **95** (2017)] I. **B**arjašić, L. **G**ulin, I. **S**.

[CQG **33** (2016)] M. **C**vitan, P. **D**ominis **P**rester, I. **S**.

Symmetry inheritance of NLE fields [PRD 95 (2017)]

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- Symmetry inheritance at points with $\mathscr{L}_{\mathfrak{F}} \neq 0$

$$\pounds_{\mathrm{K}} F_{ab} = \alpha *F_{ab} + \beta F_{ab} \ , \quad \beta = -\frac{1}{2\mathscr{L}_{\mathfrak{F}}} \pounds_{\mathrm{K}} \mathscr{L}_{\mathfrak{F}}$$

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- For $\mathscr{L} = \mathscr{L}(\mathcal{F})$ and a non-null field we have

$$\pounds_{\mathrm{K}} F_{ab} = 0$$

on the set $\{\mathscr{L}_\mathcal{F} \neq 0\} \cap \{\mathcal{F}\mathscr{L}_{\mathcal{F}\mathcal{F}} \neq 0\}$.

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- stealth examples $\to$ [I.S.: PRD **97** (2018)]

Zeroth law of BH electrodynamics

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- via Einstein field equation [Carter 1973]

$$R(K,K) \overset{H}{=} 0$$

$$E^a E_a + B^a B_a = 8\pi T(K,K)$$

$$E^a E_a - B^a B_a = K^a K_a \mathcal{F}$$

[NLE $\rightarrow$ Rasheed hep-th/9702087]

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[NLE $\to$ Rasheed hep-th/9702087]

- via bifurcation surface [Gao: PRD **68** (2003)]

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- via Frobenius condition (staticity or circularity) [I.S. 2012, 2014; I. Barjašić, L. Gulin and I.S. 2017]

$$[k,m] = 0 \ , \quad k \wedge m \wedge \mathrm{d}k = k \wedge m \wedge \mathrm{d}m = 0$$

$$k = \partial_t \ , \quad m = \partial_\varphi$$

$$i_k \pounds_m - i_m \pounds_k = i_k i_m \mathrm{d} - \mathrm{d} i_k i_m + i_{[k,m]} \quad / \ F, \, *Z$$

$$\Rightarrow \quad k \wedge m \wedge F = 0 = k \wedge m \wedge *Z$$

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- on $H[\chi]$ generated by $\chi^a = k^a + \Omega_\mathsf{H} m^a$

$$(m^a m_a) \, \chi \wedge \mathrm{d}\Phi \stackrel{H}{=} 0 \ , \quad (m^a m_a) \, \chi \wedge \mathrm{d}\Psi \stackrel{H}{=} 0$$

Smarr formula

Euler-Gibbs-Duhem relation

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A constraint between the energy E, the temperature T, the entropy S and

the rest of the pairs $\{(x_i, X^i)\}$ of the conjugate intensive/extensive thermodynamic quantities

$$E = TS + x_i X^i$$

Euler-Gibbs-Duhem relation

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- assume that the energy $E = E(S; \{X^i\})$ is a smooth homogeneous function of *degree 1*
- use the Euler's homogeneous function theorem: $kf(\mathbf{X}) = \mathbf{X} \cdot \boldsymbol{\nabla} f(\mathbf{X})$ holds for any smooth homogeneous function $f : (\mathbb{R}^n)^\times \to \mathbb{R}$ of degree k

$$\Rightarrow \quad E = \frac{\partial E}{\partial S}\, S + \frac{\partial E}{\partial X^i}\, X^i$$

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- partial derivatives can be extracted from the first law of thermodynamics

Smarr formula (1973)

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- Mass of the Kerr-Newman black hole

$$M(\mathcal{A}_{\mathsf{H}}, J, Q^2) = \left(\frac{\mathcal{A}_{\mathsf{H}}}{16\pi} + \frac{4\pi J^2}{\mathcal{A}_{\mathsf{H}}} + \frac{Q^2}{2} + \frac{\pi Q^4}{\mathcal{A}_{\mathsf{H}}} \right)^{1/2}$$

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$$M = 2T_{\mathsf{H}}S + 2\Omega_{\mathsf{H}}J + \Phi_{\mathsf{H}}Q \ , \quad S = \frac{\mathcal{A}_{\mathsf{H}}}{4}$$

Generalized NLE Smarr

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$$M_{\mathcal{S}} = -\frac{1}{8\pi}\int_{\mathcal{S}} *\mathrm{d}k\,, \quad J_{\mathcal{S}} = \frac{1}{16\pi}\int_{\mathcal{S}} *\mathrm{d}m$$

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- Bardeen-Carter-Hawking mass formula (for 4D Einstein field equation)

$$M = \frac{\kappa \mathcal{A}_\mathsf{H}}{4\pi} + 2\Omega_\mathsf{H} J - 2\int_\Sigma \left(*T(\chi) - \frac{1}{2} T *\chi \right)$$

Generalized NLE Smarr

- key idea: using $E = -i_\chi F$ and $H = i_\chi * Z$,

$$*(E \wedge *Z + H \wedge F) = 32\pi \, \mathscr{L}_{\mathfrak{F}} T^{(\mathrm{Max})}(\chi)$$

$$E \wedge *Z + H \wedge F = -\mathrm{d}(\Phi * Z + \Psi F)$$

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$$E \wedge *Z + H \wedge F = -\mathrm{d}(\Phi * Z + \Psi F)$$

- generalized Komar charges

$$Q_{\mathcal{S}} = \frac{1}{4\pi}\int_{\mathcal{S}} *Z\ , \quad P_{\mathcal{S}} = \frac{1}{4\pi}\int_{\mathcal{S}} F$$

Generalized NLE Smarr [L. Gulin and I.S. 2018]

Generalized NLE Smarr [L. Gulin and I.S. 2018]

$$M = \frac{\kappa \mathcal{A}_{\mathsf{H}}}{4\pi} + 2\Omega_{\mathsf{H}} J + \Phi_{\mathsf{H}} Q_{\mathsf{H}} + \Psi_{\mathsf{H}} P_{\mathsf{H}} + \Delta$$

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$$\Delta_\mathsf{EH} = -\frac{\alpha^2}{360 m_e^4}\,\frac{32 Q^4}{5 r_+^5} + O(\alpha^3)$$

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- ambiguity:

$$\beta = b^{\lambda} \quad \Rightarrow \quad \Delta = \frac{\mathfrak{C}}{\lambda}\,b$$

First law of BH thermodynamics

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- generalization for gravitational Chern-Simons terms [Tachikawa 2007; Bonora et al. 2010–2013; Azeyanagi, Loganayagam, Ng 2013–2017]

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- Rasheed [hep-th/9702087]

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- Zhang and Gao [1610.01237] for $\mathscr{L}(\mathfrak{F}, \{\beta_i\})$

$$\delta M = \frac{\kappa}{8\pi}\,\delta\mathcal{A} + \Omega_{\mathsf{H}}\,\delta J + \Phi_{\mathsf{H}}\,\delta Q + \Psi_{\mathsf{H}}\,\delta P + \sum_i K_i\,\delta\beta_i$$

$$K_i = \frac{1}{16\pi}\int_\Sigma \frac{\partial\mathscr{L}}{\partial\beta_i}\,{*\chi}$$

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Thank you for the attention!

This research has been supported by the

Croatian Science Foundation under the project No. 8946.