The Doubled Geometry of String Theory

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Conference on Symmetries, Geometry and Quantum Gravity
Primošten, June 18, 2018

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Context

- Consider possible geometry for "quantum gravity"
- Motivate by principles from GR and QM
- String Theory naturally lives in such geometry

Outline

1. Motivation for "Born" Geometry
2. String Theory Actions
3. Double Field Theory
4. Born Geometry & DFT
5. Para-Hermitian Geometry
6. Generalizations, T-Duality, ...

What is Born Geometry?

Born Reciprocity

- Quantum Mechanics:
  - ➡ Coordinate space and momentum space are on equal footing (can use x or p representation)
  - ➡ Commutator $[x,p] = i\hbar$ invariant under Born reciprocity

$$x \to p, \qquad p \to -x$$

Born Reciprocity

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$$x \to p, \qquad p \to -x$$

- GR: curved spacetime - Born reciprocity requires curved momentum space

[Born '38]

  - ➡ Implications for quantum gravity? Doubled geometry?

Born Geometry

- Born reciprocity realized on phase space $\mathcal{P}$ as complex structure

$$I : \begin{pmatrix} x \\ p \end{pmatrix} \to \begin{pmatrix} p \\ -x \end{pmatrix} \qquad \boxed{I^2 = -\mathbb{1}}$$

- Classical Mechanics with phase space $\mathcal{P}$
  - ➡ Poisson bracket ⇨ symplectic structure ω
- Defines a metric on curved phase space $\mathcal{H} = \omega I$

Born Geometry

- Splitting of phase space into coordinate space and momentum space via real structure

$$K : T\mathcal{P} = L \oplus \tilde{L}$$

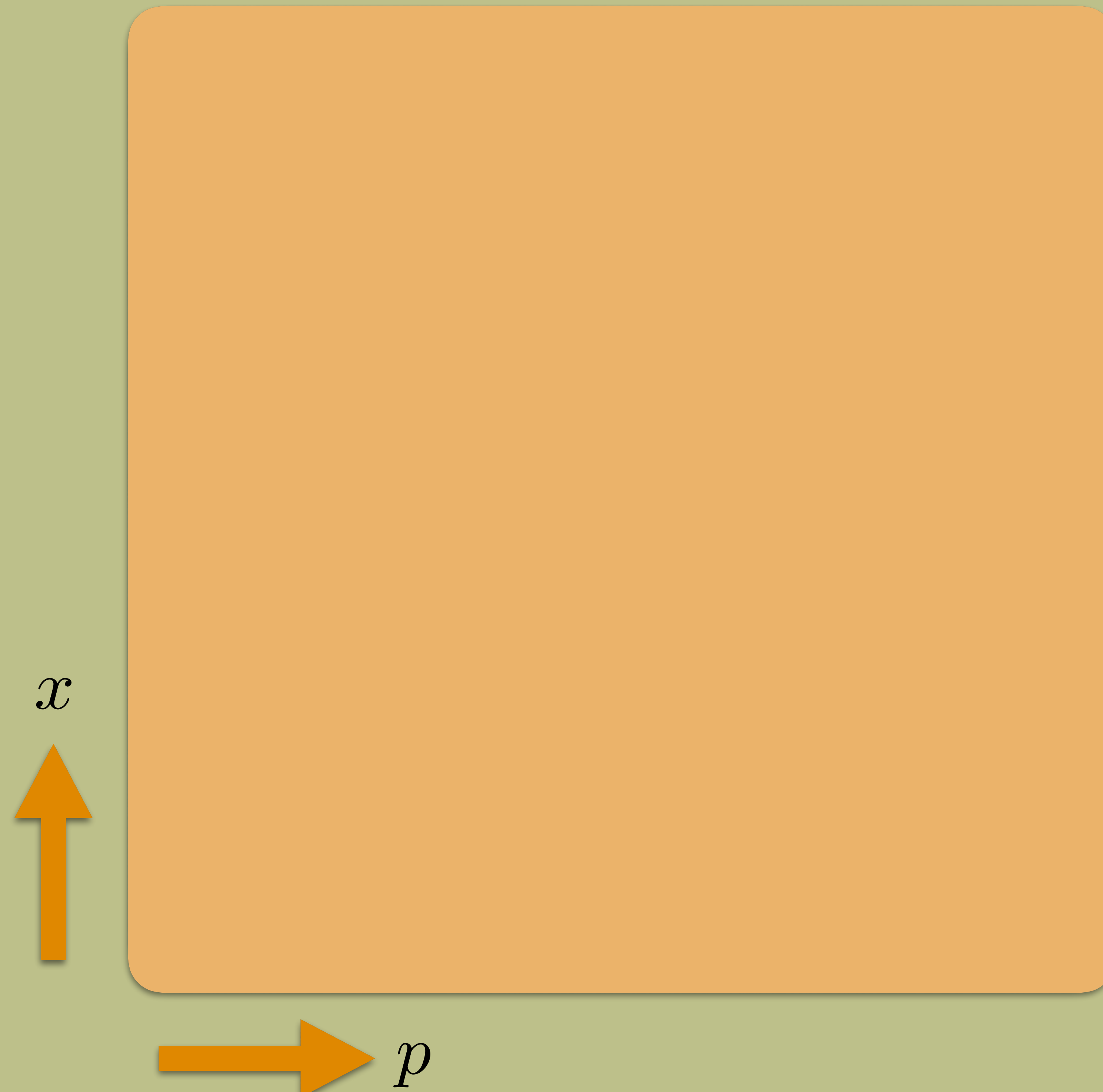
$$K^2 = +\mathbb{1}$$

$$K\Big|_L = +1 \qquad K\Big|_{\tilde{L}} = -1$$

- Defines another metric on phase space $\eta = \omega K$

The two metrics are not independent.

Doubled Space



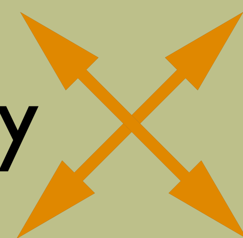
Doubled Space

- Split the doubled space

Provided by K

$$\left(\begin{array}{c|c} xx & xp \\ \hline px & pp \end{array}\right)$$

Doubled Space

- Split the doubled space
- Born reciprocity 

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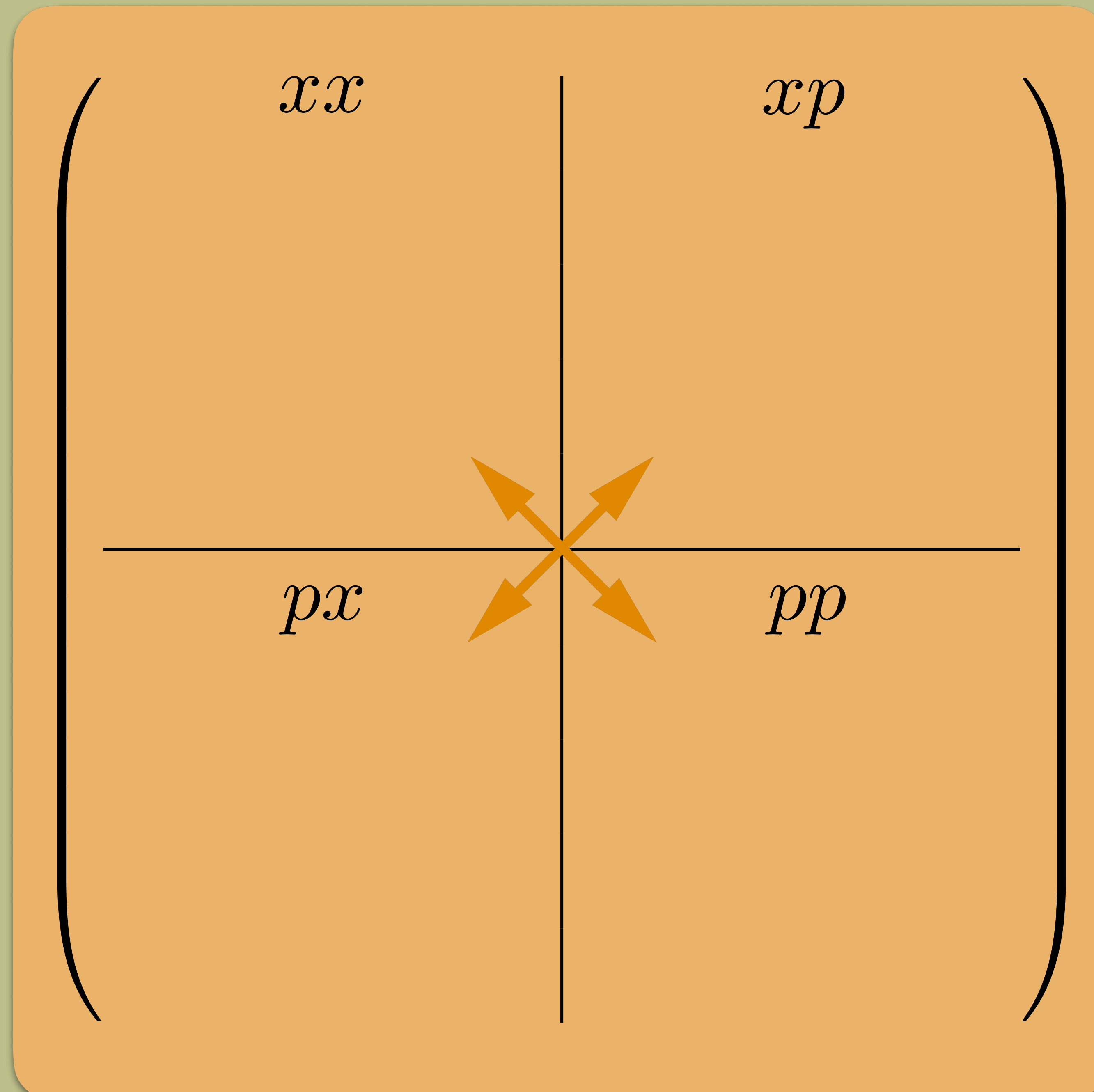


Doubled Space

- Split the doubled space
- Born reciprocity
- Structures:
  - ➡ Symplectic form ω

example of basic setup

xx $\omega_{xx} = 0$	xp $\omega_{xp} = [\,,\,]$
px $\omega_{px} = -\omega_{xp}$	pp $\omega_{pp} = 0$

Doubled Space

example of basic setup

- Split the doubled space
- Born reciprocity
- Structures:
  - ➡ Symplectic form ω
  - ➡ Neutral metric η

xx	xp
$\omega_{xx} = 0$	$\omega_{xp} = [\,,\,]$
$\eta_{xx} = 0$	$\eta_{xp} = \mathbb{1}$
px	pp
$\omega_{px} = -\omega_{xp}$	$\omega_{pp} = 0$
$\eta_{px} = \mathbb{1}$	$\eta_{pp} = 0$

Doubled Space

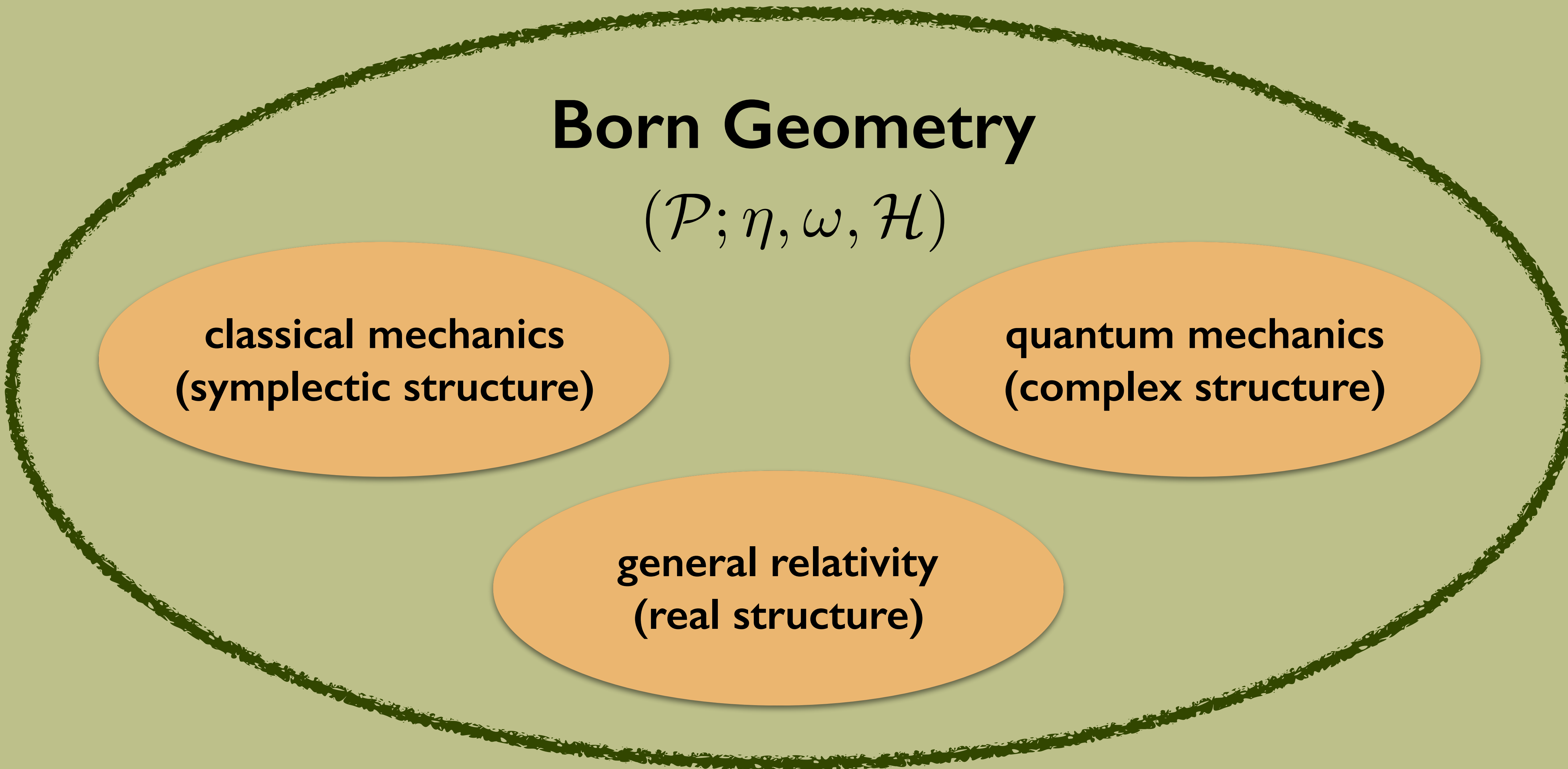
example of basic setup

- Split the doubled space
- Born reciprocity
- Structures:
  - ➡ Symplectic form ω
  - ➡ Neutral metric η
  - ➡ Dynamical metric $\mathcal{H}$

$$
\left(\begin{array}{c|c}
xx & xp \\
\omega_{xx} = 0 & \omega_{xp} = [\,,\,] \\
\eta_{xx} = 0 & \eta_{xp} = \mathbb{1} \\
\mathcal{H}_{xx} = g & \mathcal{H}_{xp} = 0 \\
\hline
px & pp \\
\omega_{px} = -\omega_{xp} & \omega_{pp} = 0 \\
\eta_{px} = \mathbb{1} & \eta_{pp} = 0 \\
\mathcal{H}_{px} = 0 & \mathcal{H}_{pp} = g^{-1}
\end{array}\right)
$$

Born Geometry

combines geometrical structures



[Freidel, Minic, Leigh '14]

Born Geometry

combines geometrical structures

Born Geometry

$$(\mathcal{P}; \eta, \omega, \mathcal{H})$$

classical mechanics
(symplectic structure)

quantum mechanics
(complex structure)

general relativity
(real structure)

Provides potential curved phase space for Quantum Gravity

[Freidel, Minic, Leigh '14]

Our goal is to show that these concepts can be found in string theory.

String Theory Actions

Sigma Models

- Map from domain Σ to target space manifold $\mathcal{T}$

$$x^\mu : \Sigma \to \mathcal{T}$$
$$\sigma^\alpha \mapsto x^\mu(\sigma)$$

- Coordinates on Σ : σ^α (worldline, worldsheet, ...)
- Coordinates on $\mathcal{T}$: $x^\mu(\sigma)$

String Theory

- String action - worldsheet $\sigma^\alpha = (\tau, \sigma)$

$$S = \int \mathrm{d}\tau \mathrm{d}\sigma \, g_{\mu\nu}(x) \partial^\alpha x^\mu \partial_\alpha x^\nu$$

➡ For now only consider background metric $g_{\mu\nu}$

➡ Scalar fields $x^\mu(\tau, \sigma)$ on string world sheet

Effective Action

Weyl anomaly cancellation!

- Quantum consistency (1-loop): $\beta = 0$

- These equations can be reproduced by (with $d = 10$)

$$S_{\text{eff}}[g(x)] = \int \mathrm{d}^d x \sqrt{g}\, R[g]$$

- Scalar fields have become coordinates of target space

- Low energy effective action of string theory
  ⇒ Supergravity

Global Symmetry

- String action Lagrangian and Hamiltonian

$$H(x,p) = \dot{x} \cdot p - L = \frac{1}{2}\mathcal{H}_{MN} Z^M Z^N$$

  - ➡ where $Z = \begin{pmatrix} x' \\ p \end{pmatrix}$

- Global $O(d,d)$ symmetry of the Hamiltonian

  - ➡ NOT a manifest symmetry of the Lagrangian!

Doubled String Action

- Can find Lagrangian with same symmetry as Hamiltonian

$$L \to L + \mathrm{d}\Omega$$

Classically equivalent

➡ $O(d,d)$ invariant string action

$$S_{\mathrm{doubled}}[g(X)] = \int \mathrm{d}\tau \mathrm{d}\sigma \, L_{\mathrm{doubled}}$$

[Tseytlin '90]

➡ Not Lorentz invariant - impose as constraint

Effective Action

[Berman, Copland, Thompson '08;
Copland '11]

- Quantum consistency (1-loop): $\beta = 0$

- Get low energy effective action for background fields

$$S_{\text{eff}}[g(X)] = \int \mathrm{d}^{2d}X\, L[g]$$

This is Double Field Theory

- Doubled coordinates: $X = \begin{pmatrix} x \\ \tilde{x} \end{pmatrix}$
  - ➡ with dual coordinate defined via $\tilde{x}' = p$

- Lorentz constraint removes dependence on half the coords.

Need the doubled formulation of the string to have manifest $O(d,d)$ symmetry.

The effective theory is DFT.

What does this doubled target space geometry look like?

Double Field Theory

Double Field Theory

[Siegel '93; Hull & Zwiebach '09]

- Formulated on a doubled space $X = \begin{pmatrix} x \\ \tilde{x} \end{pmatrix}$
- Unification of background fields in target space

$$\mathcal{H} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}$$

- Unification of local symmetries (diffeos & gauge trafos)

➡ Closure of symmetry algebra gives constraint

- Manifest global $O(d,d)$ symmetry of the effective action

DFT Action

- Doubled Target Space

$$S_{\mathrm{DFT}} = \int \mathrm{d}^{2d}X\, e^{-2\Phi}\, \mathcal{R}(\mathcal{H}, \Phi)$$

Generalized Ricci scalar

DFT Action

- Doubled Target Space

$$S_{\rm DFT} = \int \mathrm{d}^{2d}X e^{-2\Phi}\, \mathcal{R}(\mathcal{H}, \Phi)$$

Generalized Ricci scalar

- Reduction to Supergravity
  - ➡ Impose constraint as $\tilde{\partial} = 0$
  - ➡ Integrate over dual coordinates

$H = \mathrm{d}B$

$$S_{\rm SUGRA} = \int \mathrm{d}^d x \sqrt{g} e^{-2\phi} \Big[R(g) - \frac{1}{12} H^2 + 4(\partial\phi)^2 \Big]$$

Born Geometry & DFT

Doubled String Action

includes topological term

$$S_{\text{doubled}} = \frac{1}{2}\int \mathrm{d}\tau\mathrm{d}\sigma\left[(\eta_{MN}+\omega_{MN})\partial_\tau X^M\partial_\sigma X^N - \mathcal{H}_{MN}\partial_\sigma X^M\partial_\sigma X^N\right]$$

[Tseytlin '90; Giveon, Rocek '91; Hull '06]

String action containing all three structures

- Added symplectic form to ordinary worldsheet action
- Global $O(d,d)$ symmetry manifest
- Interpret target space as Born geometry

Born Geometry & DFT

- Born geometry:
  - ➡ Coordinate space and momentum space are on equal footing
  - ➡ Motivated by quantum gravity - potentially curved phase space
- Double Field Theory:
  - ➡ Effective theory for the doubled string
  - ➡ Makes symmetries of string Hamiltonian manifest in the target space

Born Geometry & DFT

[Freidel, FJR, Svoboda '17]

- What have we achieved:
  - ➡ All ingredients of Born geometry $(\eta, \omega, \mathcal{H})$ are present in the doubled string worldsheet formulation
  - ➡ Clarification of relations between DFT and generalized geometry
- What do we want to achieve:
  - ➡ Make relation between Born geometry of target space and doubled string worldsheet formulation concrete
  - ➡ Generalize kinematics & dynamics of DFT

Para-Hermitian Geometry

Para-Complex Manifold

- 2d-dimensional para-complex manifold $(\mathcal{P}, K)$
- Para-complex structure $K^2 = +\mathbb{1}$
- Eigenbundles (of equal rank)

$$K\Big|_L = +1 \qquad K\Big|_{\tilde{L}} = -1 \qquad P, \tilde{P} = \frac{1}{2}(\mathbb{1} \pm K)$$

- Splitting of tangent space $K : T\mathcal{P} = L \oplus \tilde{L}$
- Integrability of L and $\tilde{L}$ is independent

Para-Hermitian Manifold

- Include pseudo-Riemannian metric η ⇨ $(\mathcal{P}, \eta, K)$
- Skew-orthogonality: $K^{\mathsf{T}}\eta K = -\eta$
- Split signature (d, d) since eigenbundles have same rank
- Fundamental form $\omega = \eta K$ (almost symplectic)

$$\omega = \eta K = -K^{\mathsf{T}}\eta = -\omega^{\mathsf{T}}$$

- Eigenbundles L and $\tilde{L}$ are isotropic w.r.t η and ω

D-structure

[Freidel, FJR, Svoboda (to appear)]

- Differentiable structure (if K integrable): $(\mathcal{P}, \eta, K, [\![\, , \,]\!])$
- Bracket operation $[\![\, , \,]\!]$ on $T\mathcal{P}$

D-bracket

  - ➡ Generalizes the Lie bracket $[\, , \,]$
  - ➡ Compatible with metric, normalized
- Leads to generalized torsion
- Eigenbundles L and $\tilde{L}$ are Dirac structures

Canonical Connection

[Freidel, FJR, Svoboda '17]

- On para-Hermitian manifold $(\mathcal{P},\eta,\omega)$
  - ➡ Levi-Civita connection $\mathring{\nabla}$ of η
  - ➡ Define the Canonical Connection

$$\nabla^c_X = P\mathring{\nabla}_X P + \tilde{P}\mathring{\nabla}_X\tilde{P}$$

  - ➡ Preserves the eigenbundles since $\nabla^c\eta = \nabla^c\omega = 0$

Canonical D-bracket

- Unique D-bracket ⇨ Canonical D-bracket

$$\eta([\![X,Y]\!],Z)=\eta(\nabla^c_X Y-\nabla^c_Y X,Z)+\eta(\nabla^c_Z X,Y)$$

- "Canonical" since projection of the Lie bracket

$$[\![P(X),P(Y)]\!]=P([P(X),P(Y)]$$
$$[\![\tilde{P}(X),\tilde{P}(Y)]\!]=\tilde{P}([\tilde{P}(X),\tilde{P}(Y)]$$

Canonical D-bracket

Naturally related to Dorfman bracket in generalized geometry

- Unique D-bracket → Canonical D-bracket

$$\eta([\![X,Y]\!],Z) = \eta(\nabla^c_X Y - \nabla^c_Y X, Z) + \eta(\nabla^c_Z X, Y)$$

- "Canonical" since projection of the Lie bracket

$$[\![P(X),P(Y)]\!] = P([P(X),P(Y)]$$
$$[\![\tilde{P}(X),\tilde{P}(Y)]\!] = \tilde{P}([\tilde{P}(X),\tilde{P}(Y)]$$

Generalized Torsion

- Ordinary torsion $T^{\nabla}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$
  - ➡ Difference between brackets $[X,Y]^{\nabla} := \nabla_X Y - \nabla_Y X$
- Generalized torsion

$$\mathcal{T}^{\nabla}(X,Y) = [\![X,Y]\!]^{\nabla} - [\![X,Y]\!]$$

  - ➡ Defined in terms of the canonical D-bracket

Born Geometry

- Include another metric $\mathcal{H}$ ⇨ $(\mathcal{P}; \eta, \omega, \mathcal{H})$
  - ➡ Chiral structure $J = \eta^{-1}\mathcal{H}$ $\boxed{J^2 = +\mathbb{1}}$

Born Geometry

- Include another metric $\mathcal{H}$ ➡ $(\mathcal{P}; \eta, \omega, \mathcal{H})$
  - ➡ Chiral structure $J = \eta^{-1}\mathcal{H}$ $\quad$ $J^2 = +\mathbb{1}$
- All three structures together give Born geometry
  - ➡ Neutral metric η
  - ➡ Symplectic form ω
  - ➡ Dynamical metric $\mathcal{H}$

$$K = \eta^{-1}\omega \qquad I = \mathcal{H}^{-1}\omega \qquad J = \eta^{-1}\mathcal{H}$$

- All three structures are compatible

Para-Quaternionic Manifold

Born Geometry

$(\mathcal{P}; \eta, \omega, \mathcal{H})$

⇨

para-Quaternions

(I, J, K)

All mutually anti-commute:

$$IJK = -\mathbb{1}$$

$$I = JK = -KJ$$

$$J = IK = -KI$$

$$K = JI = -IJ$$

Born Connection

[Freidel, FJR, Svoboda (to appear)]

- Connection compatible with Born geometry

$$\nabla^B \eta = \nabla^B \omega = \nabla^B \mathcal{H} = 0$$

- Vanishing generalized torsion

$$\mathcal{T}^{\nabla^B} = 0$$

The Born connection exists and is unique!

(Like Levi-Civita connection for Riemannian geometry.)

Born Connection

- Lagrangian subspaces $K : T\mathcal{P} = L \oplus \tilde{L}$
- Chiral subspaces $J : T\mathcal{P} = C_+ \oplus C_-$

$$P_\pm = \frac{1}{2}(\mathbb{1} \pm J) \qquad X_\pm = P_\pm(X)$$

$$\begin{aligned}\nabla^{\mathrm{B}}_X Y = {} & [\![X_-, Y_+]\!]_+ + [\![X_+, Y_-]\!]_- \\ & + (K[\![X_+, KY_+]\!])_+ + (K([\![X_-, KY_-]\!])_-\end{aligned}$$

Generalized Kinematics & Dynamics for DFT

Generalization

- Generalization of geometry suitable for strings in a doubled space
- Kinematical structure encoded in (η, ω)
  - ➡ Neutral metric $\eta \in O(d,d)$
  - ➡ Symplectic form $\omega \in Sp(2d)$
- Dynamical d.o.f. in generalized metric $\mathcal{H}(g,B)$

Generalizations for DFT

- DFT is limit of Born geometry
- Right setup to allow for general η and ω
- Accommodate:
  - ➡ Geometric and non-geometric fluxes
  - ➡ Non-commutativity, non-associativity, non-geometry, ...

T-Duality

T-Duality

- What is T-duality?
  - ➡ Sigma model with two sets of fields x^μ and $\tilde{x}_\mu$
  - ➡ Integrate out one or the other

$$\tilde{x}' = p = \dot{x} \qquad\qquad x' = w = \dot{\tilde{x}}$$

  - ➡ Get T-duality related actions
- Canonical transformation on phase space
  - ➡ No need for compact dimensions or isometries

T-Duality

[Tseytlin '90, Duff '90, Siegel '93]

- Worldsheet symmetry (exchanges σ and τ)

$$\mathrm{d}x^{\mu} \to *\mathrm{d}x^{\mu}$$

- Target space symmetry

chiral structure

$$X \to J(X)$$

- Doubled string actions with manifest symmetry

Summary

Summary

- Born geometry - possible geometry for Quantum Gravity
- Doubled String Sigma Model & Double Field Theory
  - ➡ Actions with manifest $O(d,d)$ symmetry
  - ➡ Strong relation to principles of Born geometry
- Born connection - unique, compatible connection

Outlook:
Go beyond string theory!