Quantum deformation of spacetime symmetries in three dimensions

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based on

K.Noui, A.Perez and DP, JHEP 1110, 036 (2011), [gr-qc/1105.0439]

DP, Phys. Rev. D89, 084058 (2014), [gr-qc/1305.6714]

F. Cianfrani, J. Kowalski-Glikman, DP, G. Rosati Phys.Rev. D94, 084044 (2016), [hep-th/1606.03085]

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Can quantum groups emerge from the dynamics of quantum geometry?

If this was the case, then...

What is the fate of Poincaré symmetry at the Planck scale?

It is widely believed that quantum gravity changes dramatically the spacetime structure at small distances, allowing for fluctuations of spacetime itself

$$\text{Spacetime quantum fluctuations} \quad \xrightarrow{??} \quad \text{Deformation of classical Poincaré symmetries}$$

☞ 1 possible candidate: kappa-Poincaré

An example of infinitesimal generators of the deformed group described by a Hopf algebra

[Lukierski, Ruegg, Nowicki and Tolstoi '91] [Majid, Ruegg '94]

Has kappa-Poincaré anything to do with quantum spacetime symmetries?

Previous results where a kappa-deformation of the Poincaré group appears:

✦ A possible way relying on CS formulation was proposed in [Amelino-Camelia, Smolin, Starodubtsev '03]

✦ In 3-d quantum gravity coupled to matter fields in [Freidel, Livine '05]

➢ We are going to answer those questions in the context of Euclidean 3-dim gravity

3-dim Quantum Gravity

➢ Chern-Simons formulation

[Witten '89]:
mix of path integral and canonical quantization techniques of Riemannian theory with a positive Λ
➥ link between Jones Polynomial, Chern-Simons theory and quantum gravity

[Fock, Rosly '98] [Alekseev, Grosse, Schomerus 94] [Meusburger, Schroers 03]:
combinatorial quantization with quantum groups used as regularization scheme
➥ fundamental role played by the theory of quantum groups in the construction of 3-manifolds invariants

[Reshetikhin, Turaev '91]:
equivalence between the covariant and canonical quantization of the Chern-Simons formulation
➥ relationships between quantum gravity and the theory of knot invariants

➢ BF formulation

[Ponzano, Regge '68]:
covariant quantization performed via the spin foam approach with $\Lambda = 0$
➥ Ponzano-Regge model: partition function for a triangulated compact 3-manifold

[Turaev, Viro '92]:
q-deformed version of Ponzano-Regge model
➥ Turev-Viro state sum: covariant quantization of 3-dim gravity with $\Lambda > 0$

[Noui, Perez '05]:
LQG canonical quantization of 2+1 gravity with $\Lambda = 0$
➥ relationship between physical inner product of 2+1 gravity and spin foam amplitudes of the Ponzano-Regge model

✦ In the case $\Lambda > 0$:

there are strong motivations to the idea that, in the context of LQG, it should be possible to recover the Turaev-Viro amplitudes as the physical transition amplitudes between kinematical SU(2) spin network states of 2+1 gravity with $\Lambda > 0$

Bottom-up approach:

Implementation of the dynamics $\Rightarrow$ "Emergence" of the quantum group structure

Understanding the relationship between the Turaev-Viro invariants and quantum gravity requires the understanding the dynamical interplay between classical spin-network states and q-deformed amplitudes

Classical Riemannian general relativity in 3-d

Metric formulation

Space-time M = $\Sigma \times R$: $$\mathcal{D}[N] = \int d^2x \, N^a(x)\mathcal{D}_a(x) \,, \quad \mathcal{H}[N] = \int d^2x \, N(x)\mathcal{H}(x)$$

two generators of spacial diffeomorphisms   one generator of dynamics

$$\{\mathcal{D}[N], \mathcal{D}[M]\} = \mathcal{D}[\{N, M\}]$$
$$\{\mathcal{D}[N], \mathcal{H}[M]\} = \mathcal{D}[N^a \partial_a M]$$
$$\{\mathcal{H}[N], \mathcal{H}[M]\} = \int d^2x (N\partial_a M - M\partial_a N)\det(h) h^{ab}\mathcal{D}_b$$

If we take N, N^a to be the components of the Killing vector fields associated to the maximally symmetric solutions of Einstein eq and substitute the corresponding metric, then the algebra of diffeo and Hamiltonian constraints becomes isomorphic to the isometry algebra of the given three-dimensional space of constant scalar curvature:

$\mathfrak{isu}(2)$ for the Euclidean space E^3 (of vanishing scalar curvature)

$\mathfrak{so}(4)$ for the sphere S^3 (of positive scalar curvature)

$\mathfrak{so}(3,1)$ for the hyperbolic space H^3 (of negative scalar curvature)

Locally, all the solutions to the Einstein eq look like one of these three spaces, depending on the sign of Λ

Connection formulation

$$S[e,\omega] = \int_M \mathrm{tr}[e \wedge F(\omega) + \frac{\Lambda}{3} e \wedge e \wedge e]$$

Upon the standard 2+1 decomposition, the phase space variables are the 2-dim su(2) Lie algebra valued connection $A^i{}_a$ and its conjugate momentum $E_j{}^b = \varepsilon^{bc}\, e^k{}_c\, \eta_{jk}$. The symplectic structure is defined by

$$\{A^i_a(x), E^b_j(y)\} = \delta^b_a \delta^i_j \delta^{(2)}(x,y)$$

The variation of the action with respect to these variables leads to two sets of smeared constraints

$$G[\alpha] = \int_\Sigma \alpha^i G_i = \int_\Sigma \alpha^i D_a E^a_i = 0$$

$$C_\Lambda[N] = \int_\Sigma N_i C^i_\Lambda = \int_\Sigma N_i (F^i(A) + \frac{\Lambda}{2}\epsilon^{ijk} E_j E_k) = 0$$

✧ Constraints algebra:

$$\begin{aligned} \{C(N), C(M)\} &= \Lambda\, G([N,M]) \\ \{G(\alpha), G(\beta)\} &= G([\alpha,\beta]) \\ \{C(N), G(\alpha)\} &= C([N,\alpha]) \end{aligned}$$

One can perform a transformation

$N_i \to N^a, N$:

$N_i C^i_\Lambda = N^a \mathcal{D}_a + N\mathcal{H}$

These algebras, for various signs of Λ, have an immediate geometric interpretation as the isometric algebras of the 3-d spaces of constant scalar curvature listed above (this is particularly explicit in the Chern-Simons formulation).

➥ The algebra of gauge constraints is the algebra of local spacetime symmetries

Employ this identification on the quantum level, in order to study the symmetries of quantum de Sitter and flat Euclidean spaces

The logical steps

$$\begin{tikzcd}
\mathfrak{so}(4) \arrow[d] \\
\mathfrak{su}(2)\oplus\mathfrak{su}(2) \arrow[d, "\mathcal{Q}"] \\
\mathfrak{su}_q(2)\oplus\mathfrak{su}_{q^{-1}}(2) \arrow[d] \\
\mathfrak{so}_q(4) \arrow[d, "\Lambda\to 0"] \arrow[uuu, bend right] \\
\mathcal{P}_\kappa
\end{tikzcd}$$

- Introduction of a positive cosmological constant Λ (playing the role of an IR regulator)

- Study of the algebra of quantum constraints using LQG techniques (this is where the quantum group structure emerges) and show how the Turaev-Viro invariant can be recovered from a canonical approach

- We make an Inonu-Wigner contraction of the deformed isometry algebra sending $\Lambda \to 0$, while keeping $\hbar$ finite, in order to investigate the symmetries of quantum flat Euclidean spacetime

Physical scalar product (Λ>0)

Let us define a new non-commutative connection: $A_a^{\pm i} = A_a^i \pm \sqrt{\Lambda}\epsilon_{ab}E_i^b$

such that the Gauss and curvature constraints can be expressed as

$$C_{\Lambda}[N] = \frac{1}{2}\left(H^+[N] + H^-[N]\right)$$
$$G[N] = \frac{1}{2\sqrt{\Lambda}}\left(H^+[N] - H^-[N]\right)$$

where

$$H^{\pm}[N] \equiv \int_{\Sigma} N_i F^i(A^{\pm})$$

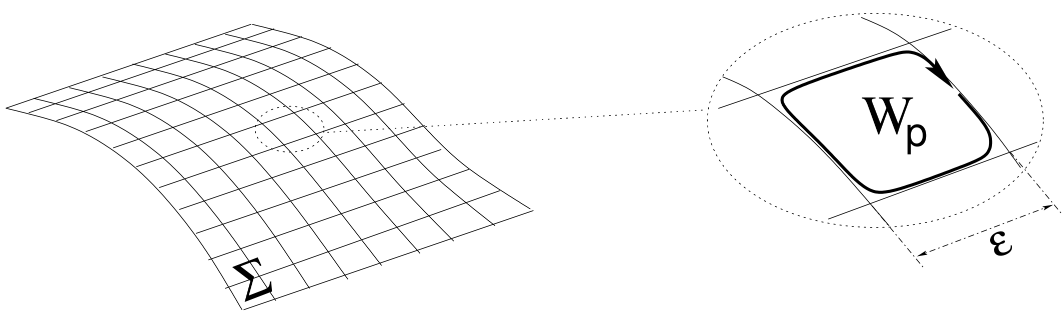
algebra of the new, equivalent set of constraints:

$$\{H^{\pm}[N], H^{\pm}[M]\} = \pm 2\sqrt{\Lambda}\ H^{\pm}[[N,M]]$$
$$\{H^+[N], H^-[M]\} = 0$$

$\mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$

local isometry

✶ Introduction of a regulator:
cellular decomposition Δ_Σ of Σ



$$\Rightarrow \quad H^{\pm}[N] = \lim_{\epsilon\to 0} \sum_{p\in\Delta_\Sigma} \mathrm{tr}\left[N_p W_p\left(A^{\pm}\right)\right] = 0$$

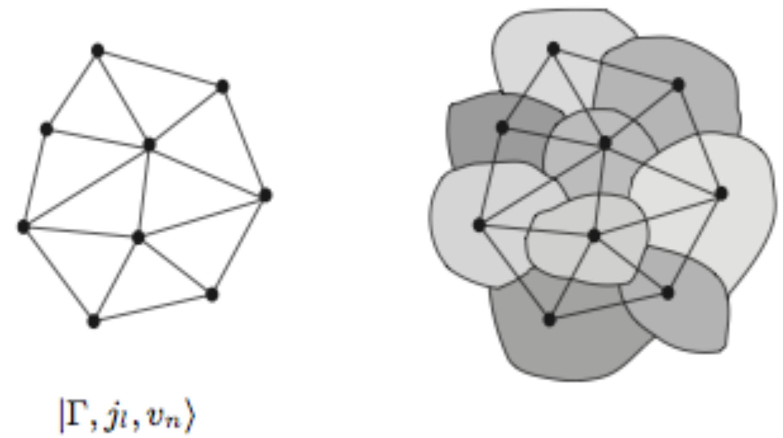
where

$$W_p(A^{\pm}) = 1 + \epsilon^2 F(A^{\pm}) + o(\epsilon^2)$$

candidate background independent regularization of the curvature constraint $C_\Lambda[N]$

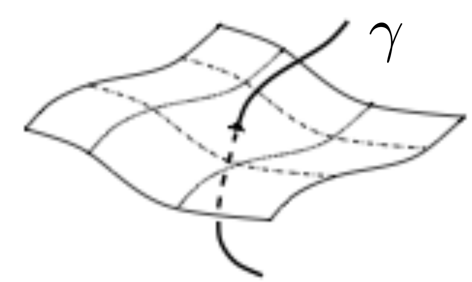
☞ quantization of the holonomy of general non-commutative A_λ

➢ Quantization of $h_\eta[A_\lambda] = Pe^{-\int_\eta A+\lambda E}$ on the kinematical Hilbert space of 2+1 LQG



- Flux of E across the curve η: $E(\eta) = \int E^a_i \tau^i n_a dt\,, \quad n_a \equiv \epsilon_{ab}\dfrac{d\eta^a}{dt}$

$$\hat{E}(\eta)\Psi_\gamma[A] = \frac{1}{2}\hbar \begin{cases} o(p)\tau_i\Psi_\gamma[A] & \text{if } \gamma \text{ ends at } \eta \\ o(p)\Psi_\gamma[A]\tau_i & \text{if } \gamma \text{ starts at } \eta \end{cases},$$



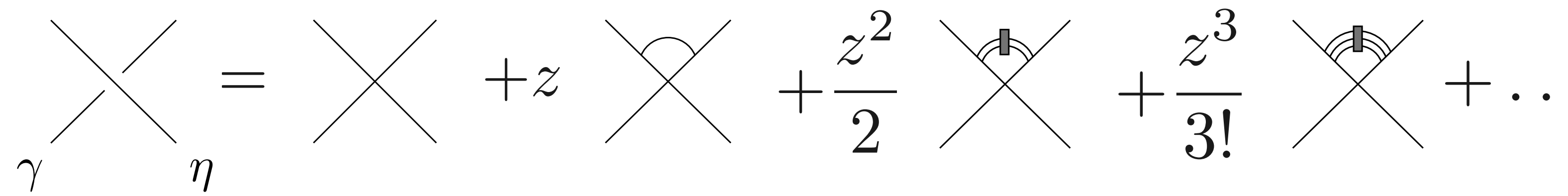
$o(p) = \pm 1$ orientation of the intersection

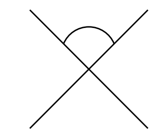
- Action on the vacuum: $\hat{h}_\eta[A_\lambda]|0\rangle = \hat{h}_\eta[A]|0\rangle = \Psi_\eta[A]$ simply creates a Wilson line excitation

- Action on a transversal Wilson line in the fundamental representation: $\hat{h}_\eta[A_\lambda]\hat{h}_\gamma[A_\lambda]|0\rangle$

quantization of each term in the series expansion of $\hat{h}_\eta[A_\lambda]$ in powers of λ ➥ quantization of products of E operators potentially ill-defined due to factor ordering ambiguities graphical notation:

$z = -io\hbar\lambda$



- First order term: E acts as LIV on γ source, as RIV on γ target  no ambiguity

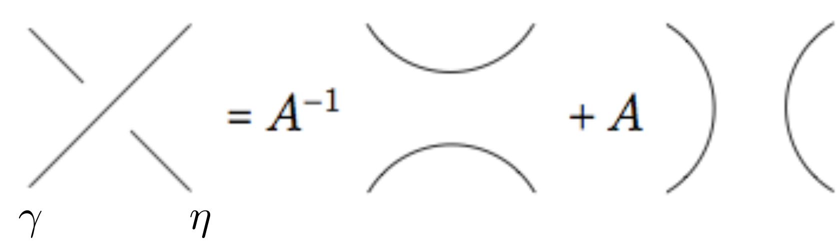
- Second order term: action of two flux operators at the same point $\longrightarrow$ Duflo map* to write

$$
\begin{aligned}
Q_D[E_jE_k] &= Q_S \circ \left(1 + \frac{1}{12}\partial_i\partial_i + \cdots\right)[E_jE_k] \\
&= \frac{1}{2}(\tau_j\tau_k + \tau_k\tau_j) + \frac{1}{6}\delta_{jk}
\end{aligned}
$$

- Third order term proportional to the first order and so on

$h_\eta(A_\lambda) \triangleright h_\gamma(A_\lambda)\,|0\rangle =$ 

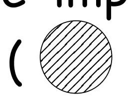
$\Rightarrow$

$h_\gamma(A_\lambda) \triangleright h_\eta(A_\lambda)\,|0\rangle =$ 

Kauffman's q-deformed binor identities for $q = A^2 = e^{\frac{i\hbar\lambda}{2}}$ [Noui, Perez, DP '11]

*See also [Freidel, Majid '07] for another application of the Duflo map in the context of 2+1 quantum gravity

- The recovering of the Kauffman bracket related to the q-deformed crossing identity is a non-trivial result since it was obtained starting from the standard SU(2) kinematical Hilbert space of LQG and combining the flux operators representation of the theory together with a mathematical input coming from the Duflo isomorphism.

- However, the full link between the role of quantum groups in 3d gravity with $\Lambda \neq 0$ and its canonical quantization can only be established if the dynamical input from the implementation of the curvature constraints is brought in: Reidermeister moves and quantum dimension ( $= -A^2 - A^{-2}$) are only to be found through dynamical considerations.

Temperley-Lieb Algebra and Recoupling Theory

Kauffman's bracket polynomial [Kauffmann, Lins '94] provides a tangle-theoretic interpretation of the Temperley-Lieb algebra and a combinatorial approach to the construction of 3-manifold topological invariants, such as the Turaev-Viro state sum model

Given an unoriented link diagram K, a state S of K is a choice of smoothing for each crossing in K, where for the smoothing there are two possibilities labelled by $A, A^{-1} \in \mathbb{C}$. Thus S appears as a disjoint set of Jordan curves in the plane decorated with labels at the site of each smoothing.

number of disjoint curves

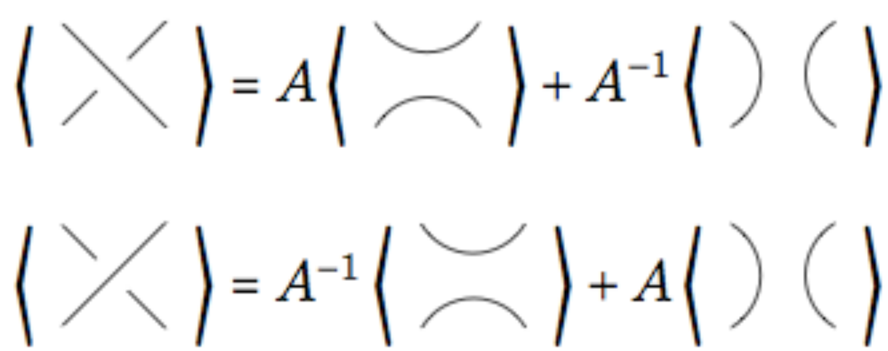
bracket polynomial: $\langle K \rangle = \sum_S \langle K|S\rangle d^{||S||}, \qquad d = -A^2 - A^{-2}$

smoothing possibilities

product of the state labels of S

satisfying the following properties:

(i)

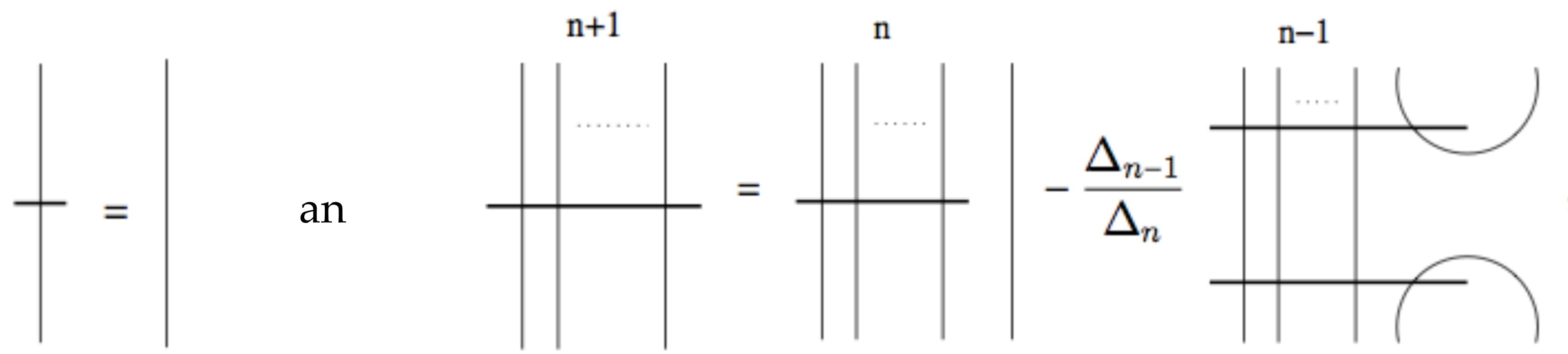


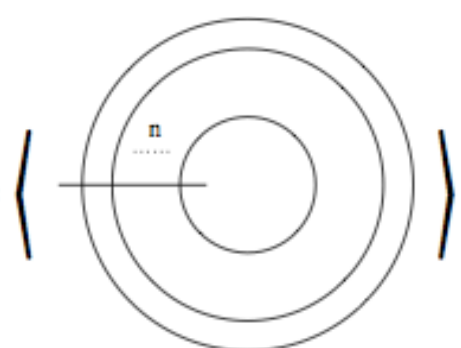
(ii) $\langle \bigcirc \sqcup K \rangle = d\, \langle K \rangle$

Properties (i) and (ii) are called Kauffman brackets and they guarantees that the bracket polynomial is an invariant of regular isotopy of link diagrams, i.e. it satisfies the Reidemeister moves of type II and III (plus underlying graphical changes induced by homeomorphisms of the plane)

Chromatic evaluation = standard way to apply the Temperley-Lieb recoupling theory to trivalent spin networks evaluations

a link of color n (with n = 2j, j being the spin Irrep associated with the link) represents n parallel lines and a symmetrizer, or projector operator



where $\Delta_n = tr(f_n) = $ 

$$= (-1)^n (n+1) \qquad SU(2)$$

$$= (-1)^n \left(\frac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}} \right) \qquad SU_q(2)$$

Definition of the trace of an n tangle as the bracket polynomial

$$\Delta_0 = 1\,, \qquad \Delta_1 = d\,, \qquad \Delta_{n+1} = d\Delta_n - \Delta_{n-1}$$

$q = A^2$ is a 2r-th primitive root of unity, i.e. $q^r = -1$, $q = exp\ i\pi/r$

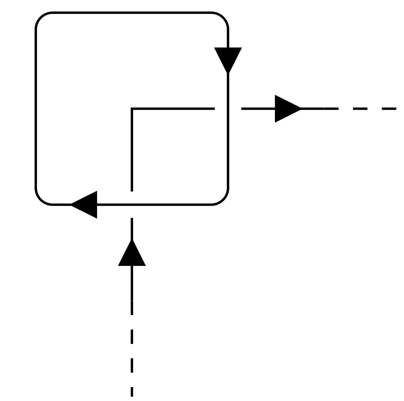
Since a trivalent spin network is obtained by joining several trivalent vertices by their edges, through this construction, a trivalent spin network determines a closed tangle. One can now use the Kauffman bracket relations (i)-(ii) for the (chromatic) evaluation of this tangle.

➡ Contractions of intertwiners can therefore be computed as chromatic evaluations of colored diagrams

Constraint Algebra

In order to be able to implement the dynamics correctly, we first need to make sure that the quantum constraint algebra properly closes. From the classical algebra we see that this amounts to having the action of the commutator of any two operators $\hat{H}^{\pm}[N_p] = \mathrm{tr}[N_p \hat{W}_p(A^{\pm})]$ to vanish on a gauge invariant state

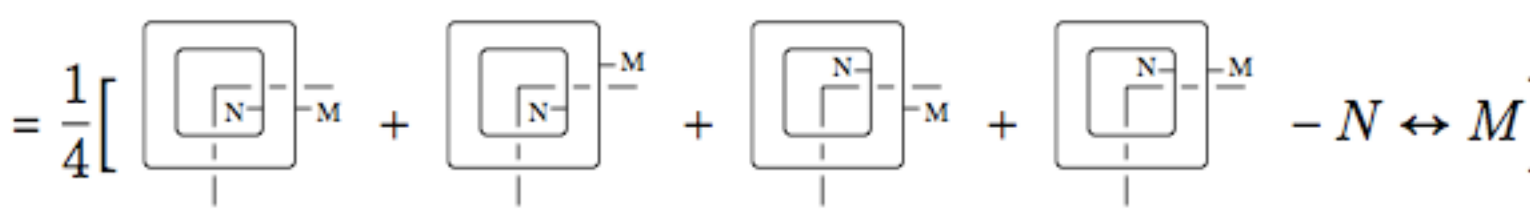
✧ we need to introduce also the dual complex Δ_{Σ^*} with plaquettes $p^* \in \Delta_{\Sigma^*}$ dual to $p \in \Delta_\Sigma$:

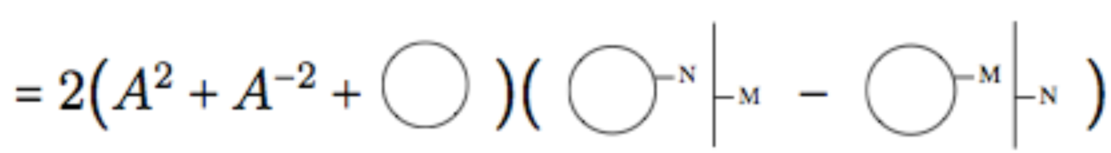


$Cyl(\Delta_{\Sigma^*}) \subset Cyl$

with this regularization, one can see that the commutator between Wilson loops on different plaquettes gives immediately zero; therefore, we only need to consider the case of two operators defined on the same plaquette.

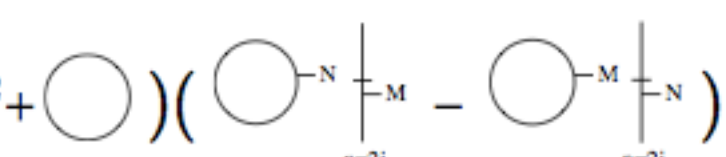
$$\left[\hat{H}^{+}[N_p], \hat{H}^{+}[M_p]\right] \triangleright |\ulcorner_{\frac{1}{2}}\rangle$$





a similar calculation for the state in a generic spin-j representation leads to

$$\left[\hat{H}^{+}[N_p], \hat{H}^{+}[M_p]\right] \triangleright |\ulcorner_j\rangle = 2(A^2 + A^{-2} +$$



➥ the commutator on a gauge invariant state doesn't vanish unless the infinitesimal loop evaluates to the spin-1/2 quantum dimension $\bigcirc_{1/2} = -(A^2 + A^{-2})$

In analogy to the Λ=0 case:

background independence and anomaly-free quantum constraints algebra ➥ definition of a physical scalar product by means of a projector operator into the kernel of $C_0[N]$: path integral representation of the theory from the canonical picture

✧ Projection operator:

$$P = \text{``}\prod_{x\in\Sigma}\delta(\hat{F}(A(x)))\text{''} = \int D[N]exp\left(i\int_\Sigma Tr[N\hat{F}(A)]\right)$$

✧ Physical scalar product:

$$\begin{aligned} < s,\ s' >_{ph} =< Ps,\ s' > \quad &:= \quad \lim_{\epsilon\to 0} \quad < \prod_p \delta(W_p)s,\ s' > \\ &= \quad \lim_{\epsilon\to 0} \quad \sum_{j_p}(2j_p+1) < \prod_p \chi_{j_p}(W_p)\ s,\ s' > \end{aligned}$$

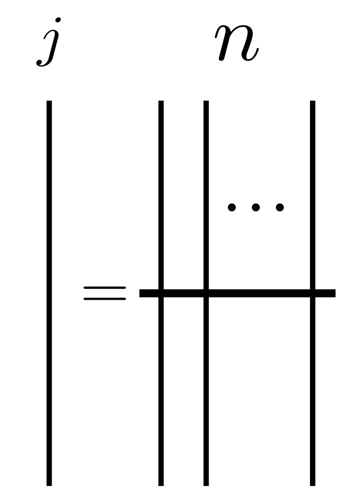
➥ Ponzano-Regge amplitudes

Physical Transition Amplitudes

✧ Physical scalar product:

$$\langle s,\ s'\rangle_{ph-\Lambda} = \langle P^{\Lambda} s,\ s'\rangle := \lim_{\epsilon\to 0}\ \langle \prod_p\ \delta(W_p^{+}) s,\ s'\rangle$$

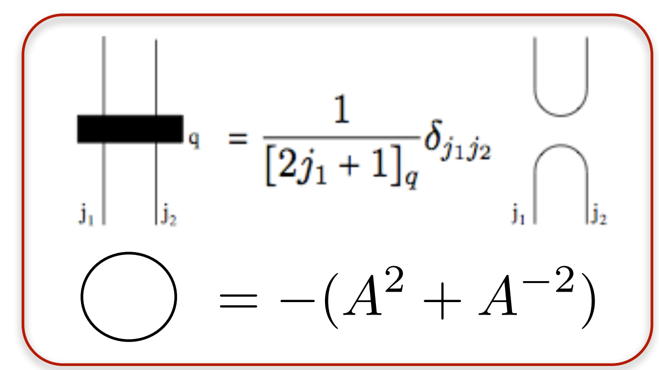
$$= \lim_{\epsilon\to 0}\ \sum_{j_p} [2j_p+1]_q \langle \prod_p \chi_{j_p}(W_p^{+})\ s,\ s'\rangle$$

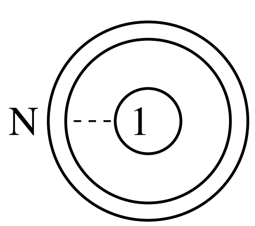


Physical amplitudes between classical kinematical spin network states:

replace every link-j in s, s′ with a corresponding n-symmetrizer (n = 2j) and, by correctly joining all the strands at each intertwiner, the two closed spin network graphs associated to the states s, s′ can then be expressed as a combination of products of loops

➥ To recover the bracket polynomial we simply need to show that the physical transition amplitude between products of loops is equal to the products of the quantum dimensions in the spin-j representations coloring the respective loops

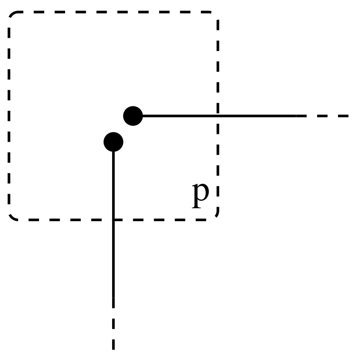


$\langle P^{\Lambda}\emptyset,$  $\rangle = \lim_{\epsilon\to 0}\int \left(\prod_h dg_h\right)\prod_n \chi_{k_n}(g_n)\prod_p\sum_{j_p}[2j_p+1]_q\ \chi_{j_p}(W_p^{+}) = \prod_{n=1}^{N}[2k_n+1]_q$

➥ Turaev-Viro amplitudes

The R-matrix

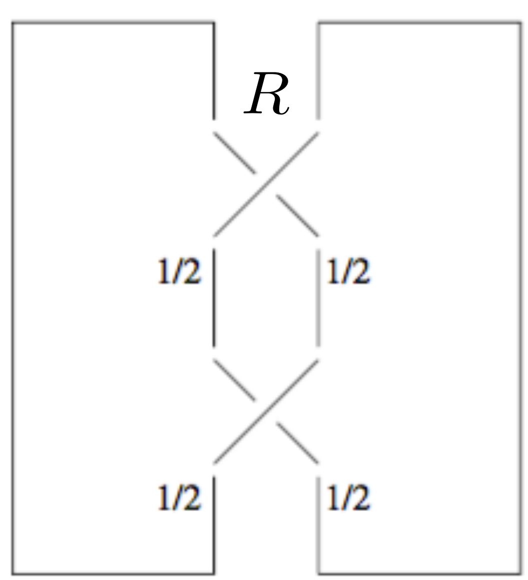
Not gauge-invariant state

$|\Psi\rangle =$ 

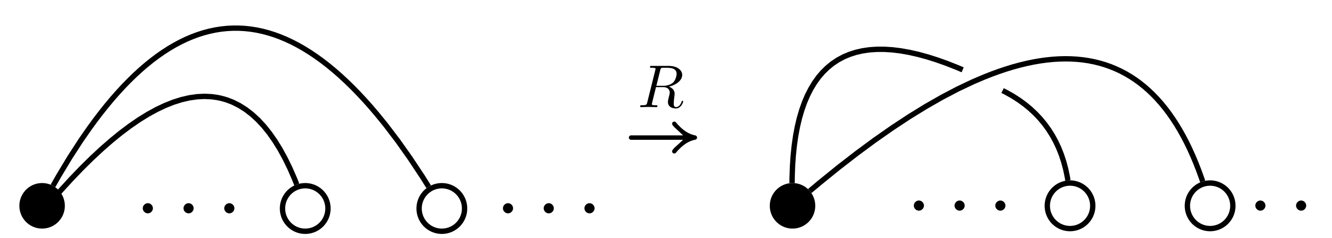
$$\left[\hat{H}^{\pm}[N_p], \hat{H}^{\pm}[M_p]\right]|\Psi\rangle = \pm\Lambda\left(A^2 + A^{-2}\right)\ \hat{H}^{\pm}[[N_p, M_p]]|\Psi\rangle\,,$$
$$\left[\hat{H}^{\pm}[N_p], \hat{H}^{\mp}[M_p]\right]|\Psi\rangle = 0$$

At the quantum level the algebra of constraints is deformed. We know already from the q-deformed skein relations that the new symmetry replacing the classical su(2) one is the quantum group $SL_q(2)$

To unravel the quantum group symmetry encoded in the constraint algebra of 2+1 LQG, the relevant structure to look at is the R-matrix structure behind the crossing properties of two non-commutative holonomies defining the constraints. We want to show explicitly that the q-deformed crossing identity can be represented in terms of the $SL_q(2)$ R-matrix



✶ Particles picture: The action of the R-matrix represents an isometry of the physical Hilbert space of gravity coupled to point particles



Since each link-1/2 carries a representation vector space V = C^2, we can derive the form of the R-matrix associated to the generators $\hat{H}^{+}[N_p]$ by studying the action of the crossing operators on the tensor product vector space $V \otimes V$

 $: V \otimes V \to V \otimes V$

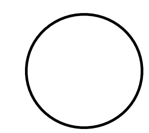
 $: V \otimes V \to V \otimes V$

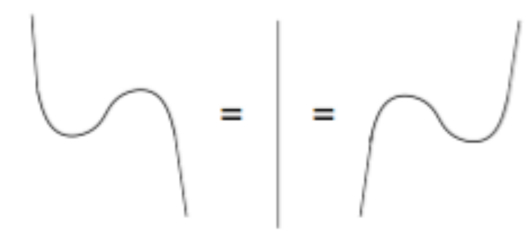
 $: V \otimes V \to V \otimes V$

 $: V \otimes V \to \mathbb{C}$

 $: \mathbb{C} \to V \otimes V$.

Given an orthonormal basis of V = C^2 formed by the vectors v_1 , v_2 , we demand compatibility with

 $= -(A^2 + A^{-2})$



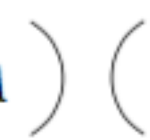
 $: \quad V \otimes V \to \mathbb{C}$

$$\begin{aligned} v_1 \otimes v_1 &\to 0 \\ v_1 \otimes v_2 &\to A \\ v_2 \otimes v_1 &\to -A^{-1} \\ v_2 \otimes v_2 &\to 0 \end{aligned}$$

 $: \quad \mathbb{C} \to V \otimes V$

$$1 \to -Av_1 \otimes v_2 + A^{-1} v_2 \otimes v_1$$

Action of the crossing :

 $= A$  $+ A^{-1}$  $: V \otimes V \to V \otimes V$

$$\begin{aligned} v_1 \otimes v_1 &\to Av_1 \otimes v_1 \\ v_1 \otimes v_2 &\to A^{-1} v_2 \otimes v_1 \\ v_2 \otimes v_1 &\to A^{-1} v_1 \otimes v_2 + A^{-1}(A^2 - A^{-2}) v_2 \otimes v_1 \\ v_2 \otimes v_2 &\to Av_2 \otimes v_2 \end{aligned}$$

We now want to show that the action above of the crossing operator corresponds exactly to the action of the $SL_q(2)$ R-matrix in the spin-1/2 (2-dim) representation on $C^2 \otimes C^2$

Let $q = e^h$ and let $SL_q(2)$ be the algebra generated by X_+, X_-, e^{hH} with relations

$$X_+X_- - X_-X_+ = \frac{e^{2hH} - e^{-2hH}}{q - q^{-1}}\,, \qquad e^{hH}X_+ = qX_+e^{hH}\,, \qquad e^{hH}X_- = q^{-1}X_-e^{hH}$$

We then obtain a bi-algebra given the co-products

$$\Delta X_+ = X_+ \otimes e^{hH} + e^{-hH} \otimes X_+$$
$$\Delta X_- = X_- \otimes e^{-hH} + e^{hH} \otimes X_-$$
$$\Delta e^{hH} = e^{hH} \otimes e^{hH}$$

and co-units

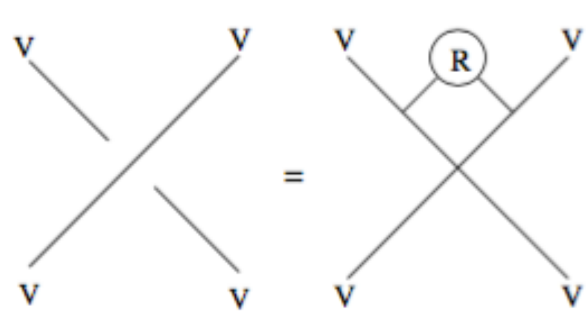
$$e(X_+) = e(X_-) = 0\,, \qquad e(e^{hH}) = 1$$

This gives a quasi-triangular bi-algebra with $R \in SL_q(2) \otimes SL_q(2)$ given by

$$R = \sum_{n=0}^{\infty} \frac{q^{\frac{n}{2}(n+1)}(1-q^{-2})^n}{[n]_q!} e^{2h(H\otimes H)} X_+^n \otimes X_-^n \qquad \text{where} \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

If we now use the 2-dim representation ρ of $SL_q(2)$, in which X_+, X_-, H act as linear transformations on C^2 to compute

$$\rho(R)(v \otimes v') = \sum_{ij} R^{ij} S(\rho(G_i)v \otimes \rho(G_j)v') = \sum_{ij} R^{ij} \rho(G_j)v' \otimes \rho(G_i)v \quad \in V' \otimes V$$

➥ 

for $\quad A = e^{\frac{h}{2}}\,, \quad h = i\hbar\frac{\sqrt{\Lambda}}{2}$

In the case of $\hat{H}_-[N_p]$, one recovers the $SL_q(2)$ R-matrix for

$$h = -i\hbar\frac{\sqrt{\Lambda}}{2}$$

From SLq(2) to SOq(4)

We have explicitly shown how the introduction of a regulator, in the form of a discrete structure, required by the LQG quantization scheme leads to a quantum deformation of the local su(2)⊕su(2) symmetry generated by the classical constraint algebra. At the quantum level, the local isometry becomes $SL_q(2) \oplus SL_{-q}(2)$.

We started with the classical algebra:

$$[B_a,\, B_b] = \Lambda\, \epsilon_{ab}{}^{c}\, R_c$$
$$[R_a,\, R_b] = \epsilon_{ab}{}^{c}\, R_c$$
$$[B_a,\, R_b] = \epsilon_{ab}{}^{c}\, B_c$$

$\Rightarrow$

with the definition $A^{\pm}_a = \frac{1}{2}(R_a \pm \frac{1}{\sqrt{\Lambda}} B_a)$

$$[A^{\pm}_a,\, A^{\pm}_b] = \epsilon_{ab}{}^{c}\, A^{\pm}_c$$
$$[A^{\pm}_a,\, A^{\mp}_b] = 0$$

$$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \simeq \mathfrak{so}(4)$$

$\Rightarrow$

$$i = 1, 2$$
$$[H_i,\, H_j] = 0$$
$$[H_i,\, X_{\pm j}] = \pm\delta_{ij}\, X_{\pm j}$$
$$[X_{+i},\, X_{-j}] = \delta_{ij}\, H_j$$

In the quantum theory:

$$[H_i, H_j] = 0,$$
$$[H_i, X_{\pm j}] = \pm\delta_{ij} X_{\pm j},$$
$$[X_{+i}, X_{-j}] = \delta_{ij} \frac{\sinh(2h_i H_i)}{\sinh(h_i)}$$

where $h_1 = -h_2 = h = iz\,, \quad z = \sqrt{\Lambda}/\kappa$

with $\kappa = 2/\hbar$

$\Rightarrow$

$$[E, P_i] = -\Lambda N_i,$$
$$[P_1, P_2] = \Lambda \frac{\sinh(zM)}{\sin(z)} \cosh\left(zE/\sqrt{\Lambda}\right),$$
$$[N_i, E] = -P_i,$$
$$[N_i, P_j] = \delta_{ij} \sqrt{\Lambda} \frac{\sinh\left(zE/\sqrt{\Lambda}\right)}{\sin(z)} \cosh(zM),$$
$$[N_1, N_2] = \frac{\sinh(zM)}{\sin(z)} \cosh\left(zE/\sqrt{\Lambda}\right),$$
$$[M, N_i] = \epsilon_i{}^{j} N_j, \quad [M, P_i] = \epsilon_i{}^{j} P_j, \quad [M, E] = 0$$

$\mathfrak{so}_q(4)$ translations, boosts and rotation in 3D

$$N_1 = B_1,$$
$$N_2 = B_2,$$
$$E = \sqrt{\Lambda} B_3,$$
$$P_2 = -\sqrt{\Lambda} R_1,$$
$$P_1 = \sqrt{\Lambda} R_2,$$
$$M = R_3$$

From $SO_q(4)$ to κ-Poincaré

Having obtained the deformed symmetry algebra for the case of the Euclidean de Sitter quantum gravity in 3D, we now want to make the contraction $\Lambda \to 0$, to obtain a symmetry that replaces the standard Poincaré symmetry in the case of quantum spacetime.

To do this, we perform the Inonu-Wigner contraction by taking the limit $\sqrt{\Lambda}, z \to 0$ while keeping $z/\sqrt{\Lambda} = \hbar/2$ finite.

We get:

Algebra

$$[E, P_i] = [P_1, P_2] = 0,$$
$$[N_i, E] = -P_i,$$
$$[N_i, P_j] = \delta_{ij}\kappa \sinh(E/\kappa),$$
$$[N_1, N_2] = M \cosh(E/\kappa),$$
$$[M, N_i] = \epsilon_i^{\ j} N_j, \qquad [M, P_i] = \epsilon_i^{\ j} P_j, \qquad [M, E] = 0$$

Antipodes

$$S(M) = -M, \qquad S(E) = -E,$$
$$S(P_i) = -P_i, \qquad S(N_i) = -N_i - \frac{1}{\kappa} P_i$$

Co-products

$$\Delta E = E \otimes 1 + 1 \otimes E,$$
$$\Delta M = M \otimes 1 + 1 \otimes M,$$
$$\Delta P_i = P_i \otimes e^{\frac{1}{2}E/\kappa} + e^{-\frac{1}{2}E/\kappa} \otimes P_i$$
$$\Delta N_1 = N_1 \otimes e^{\frac{1}{2}E/\kappa} + e^{-\frac{1}{2}E/\kappa} \otimes N_1 - \frac{1}{2\kappa} P_2 \otimes e^{\frac{1}{2}E/\kappa} M + \frac{1}{2\kappa} e^{-\frac{1}{2}E/\kappa} M \otimes P_2$$
$$\Delta N_2 = N_2 \otimes e^{\frac{1}{2}E/\kappa} + e^{-\frac{1}{2}E/\kappa} \otimes N_2 + \frac{1}{2\kappa} P_1 \otimes e^{\frac{1}{2}E/\kappa} M - \frac{1}{2\kappa} e^{-\frac{1}{2}E/\kappa} M \otimes P_1$$

(2+1)-D κ-Poincaré algebra in standard basis and Euclidean signature

If we had $h_1 = h_2 = h$, the co-products of the boosts N_i would be diverging

Concluding remarks

➢ We have shown that the local isometry so(4) ≃ su(2) ⊕ su(2) of classical 3-d gravity with a positive cosmological constant is deformed at the quantum level, where the space-time local symmetry becomes the quantum group $so_q(4)$. It is remarkable that the primary reason for the emergence of the deformed Hopf structure in the theory is anomaly cancellation.

➢ The physical scalar product of the theory is a straightforward generalization of the $\Lambda = 0$ case and we have shown how this allows us to recover the Turaev-Viro state-sum amplitudes. This represents a highly non-trivial test for the loop approach to quantum gravity, showing complete agreement with other well defined quantization schemes.

➢ By making the contraction $\Lambda \to 0$, the symmetry algebra of flat quantum Euclidean spacetime in 3D turns out to be the κ-Poincaré algebra.

Relevance for the 4D case: In the presence of an inner boundary, the boundary theory is effectively a 3D theory. In certain cases, the phase space and the set of constraints can resemble those of 2+1 gravity with Λ, e.g. for a spherically symmetric isolated horizon

⇨ Horizon entropy with LQG methods [Sahlmann, DP '15]