One-loop renormalisation of κ-Poincaré invariant field theories

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Motivations

- According to many theoretical studies, the classical description of space-time, *as a continuum*, might be no longer adequate to reconcile gravity with quantum mechanics at very high energy (~ Planck scale).

- Instead, a more appropriate description could be provided by the data of a *non-commutative algebra of coordinate operators* (replacing the usual commutative local coordinates on smooth manifold).

- In this spirit, the κ-*Minkowski space* appears in the physics literature to be one of the most studied Lie algebra type non-commutative space-time

$$[x_0, x_i] = i\kappa^{-1} x_i, \ \left[x_i, x_j\right] = 0, \ i, j = 1, \dots, 3,$$

where the deformation parameter is *dimensionful*, $[\kappa] = \mathrm{L}^{-1}$ (natural units).
→ κ-Poincaré sym. & κ ~ Planck mass or some intermediate (QG) scale.
→ Various applications e.g. in QG models, DSR, relative locality…

- Non-commutative space-time ⇒ Non-Commutative Field Theory.

Presentation of the κ-Minkowski space (see J.-C. Wallet)

- A convenient presentation (for studying NCFT) of the κ-Minkowski space is provided by the combination of harmonic analysis on the locally-compact non-unimodular Lie group $\mathrm{G} \coloneqq \mathbb{R}^3 \rtimes \mathbb{R}$ with the *Weyl quantisation scheme*.

- This procedure leads to the following star product

$$(f \star g)(x) = \int dpdy\, e^{-ipy} f(x_0 + y, \vec{x}) g\left(x_0, e^{-p/\kappa}\vec{x}\right),$$

while a natural involution on κ-Minkowski is given by

$$f^\dagger(x) = \int dpdy\, e^{-ipy} \bar{f}\left(x_0 + y, e^{-p/\kappa}\vec{x}\right).$$

✦ Remarks: $f \in \mathcal{M}_\kappa$ is $\mathbb{C}$-valued (smooth function), but $f^\dagger \neq \bar{f}$.
Here, $\mathcal{M}_\kappa$ denotes the non-commutative algebra of fields modeling κ-Minkowski.

Construction of κ-Poincaré invariant action functional

- The aforementioned presentation of the κ-Minkowski space provides us with (almost) all the needed material for constructing an action functional $\mathcal{S}_{\kappa,\star}$ aiming to encode the dynamics of an interacting complex scalar field on κ-Minkowski background.

- It is convenient to introduce a Hilbert product on $\mathcal{M}_\kappa$

$$\langle f, g \rangle_\star \coloneqq \int d^4x \left(f \star g^\dagger\right)(x) = \int d^4x\, f(x)\bar{g}(x).$$

$\Rightarrow \langle f, f \rangle_\star \in \mathbb{R}$ and $\langle f, Kf \rangle_\star \in \mathbb{R}$, $\forall f \in \mathcal{M}_\kappa$ and $K: \mathcal{M}_\kappa \to \mathcal{M}_\kappa$ self-adjoint.

- We further require

  a) $\mathcal{S}_{\kappa,\star}$ to be κ-Poincaré invariant

  $$\Rightarrow \mathcal{S}_{\kappa,\star}\left[\phi, \phi^\dagger\right] = \int d^4x\, \mathcal{L}\left[\phi, \phi^\dagger\right](x),\ \ \mathcal{L}\left[\phi, \phi^\dagger\right] \in \mathcal{M}_\kappa;$$

  b) $$\lim_{\kappa\to\infty} S_{\kappa,\star}\left[\phi, \phi^\dagger\right] = \int d^4x \left(\tfrac{1}{2}\bar{\phi}(x)(-\partial_\mu\partial^\mu + m^2)\phi(x) + \tfrac{g}{4!}|\phi(x)|^4\right).$$

Kinetic term(s)

- We assume $\mathcal{S}_{\kappa,\star}[\phi,\phi^\dagger] = \mathcal{S}_{\kappa,\star}^{kin}[\phi,\phi^\dagger] + \mathcal{S}_{\kappa,\star}^{int}[\phi,\phi^\dagger]$.

$$\Rightarrow \mathcal{S}_{\kappa,\star}^{kin}[\phi,\phi^\dagger] \coloneqq \tfrac{1}{4}\langle\phi, T_1\phi\rangle_\star + \tfrac{1}{4}\langle\phi^\dagger, T_2\phi^\dagger\rangle_\star,$$

where $T_i \coloneqq K + m_i^2$ is a self-adjoint kinetic operator K with dense domain in $\mathcal{M}_\kappa$ possibly supplemented by a mass-like term $m_i \in \mathbb{R}$.

- Further assuming K to be a differential operator, one can identify

$$K(\partial) \to K(P),\, P_\mu = -i\partial_\mu.$$

$$\Rightarrow \mathcal{S}_{\kappa,\star}^{kin}[\phi,\phi^\dagger] = \tfrac{1}{4}\big\langle\big(T_1 + \sigma S(T_2)\big)\phi, \phi\big\rangle_\star,$$

where $\sigma \coloneqq e^{3i\partial_0/\kappa}$ is the twist characterising the twisted trace property

$$\int d^4x\,(f \star g)(x) = \int d^4x\,(\sigma g \star f)(x),$$

and S is the antipode of the κ-Poincaré algebra s.t. $(h \triangleright f)^\dagger = S(h)^\dagger \triangleright f$.

- A first natural choice for the kinetic operator is provided by the *first Casimir operator* of the κ-Poincaré algebra which is given in the Majid-Ruegg basis by

$$K_c(P) \equiv C_\kappa(P) \coloneqq 4\kappa^2 \sinh^2\left(\frac{P_0}{2\kappa}\right) + e^{P_0/\kappa}\vec{P}^2.$$

- A second natural choice is provided by the *square of an equivariant Dirac operator* which can be written has

$$K_{eq}(P) \coloneqq \mathrm{C}_\kappa(\mathrm{P}) + \frac{\mathrm{C}_\kappa^2(\mathrm{P})}{4\kappa^2}.$$

- In both cases, $K \underset{\kappa\to\infty}{\longrightarrow} \partial_\mu\partial^\mu$ and one has $S(K) = K$ s.t.

$$\mathcal{S}_{\kappa,\star}^{kin}[\phi,\phi^\dagger] = \tfrac{1}{4}\langle(1+\sigma)(K+M^2)\phi,\phi\rangle_\star,$$

$$\text{s.t. } \min(m_1^2,m_2^2) \leq M^2(m_1,m_2) \coloneqq \tfrac{m_1^2+m_2^2\sigma}{1+\sigma} \leq \max(m_1^2,m_2^2)\,.$$

$\Rightarrow$ We restrict our attention to the case $M(m_1,m_2) = m$, constant.

Interaction term(s)

- According to the terminology of NCFT, one distinguishes two *orientable interactions*

$$\mathcal{S}_{\kappa,\star}^{int,o}[\phi,\phi^\dagger] := \frac{g}{4!}\Big(\big\langle\phi^\dagger\star\phi,\phi^\dagger\star\phi\big\rangle_\star + \big\langle\phi\star\phi^\dagger,\phi\star\phi^\dagger\big\rangle_\star\Big),$$

where the ϕ's alternate with the $\phi^\dagger$'s and two *non-orientable interactions*

$$\mathcal{S}_{\kappa,\star}^{int,no}[\phi,\phi^\dagger] := \frac{g}{4!}\Big(\langle\phi\star\phi,\phi\star\phi\rangle_\star + \big\langle\phi^\dagger\star\phi^\dagger,\phi^\dagger\star\phi^\dagger\big\rangle_\star\Big),$$

with dimensionless coupling constant $g \in \mathbb{R}$.

⇒ The usual single interaction potential for the (commutative) $|\phi|^4$-model splits into 4 different interactions at the non-commutative level. (This results from the non-commutativity & the twisted trace property of the Lebesgue integral.)

NCFT as ordinary non-local complex scalar field theories

- A key point for deriving "easily" the loop corrections to the n-point functions is to notice that the NCFT involving the star product can actually be interpreted as an ordinary, albeit non-local, complex scalar field theory (involving pointwise product).

- Mere combination of the integral expressions for the star product and the involution with the expression for $S_{\kappa,\star}$ yields the identification

$$S_{\kappa,\star}[\phi,\phi^\dagger] \to S_\kappa[\phi,\bar{\phi}\,].$$

The kinetic term for the non-local QFT is given by

$$S_\kappa^{kin}[\phi,\bar{\phi}\,] \coloneqq \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\,\bar{\phi}(\mathrm{k})\mathcal{K}(k)\phi(k),$$

$$\mathcal{K}(k) \coloneqq \tfrac{1}{2}\big(1+e^{-3k^0/\kappa}\big)\big(K(k)+m^2\big)$$

$\Rightarrow$ Straightforward derivation of the propagator

$$\Delta(k) = 1/\mathcal{K}(k).$$

- The interaction term is now given by

$$S_\kappa^{int}[\phi,\bar{\phi}\,] := \frac{g}{4!}\int \prod_{\ell=1}^{4} \frac{d^4k_\ell}{(2\pi)^4}\ \bar{\phi}(k_1)\phi(k_2)\bar{\phi}(k_3)\phi(k_4)\,\mathcal{V}_{1234},$$

with, for the orientable model

$$\mathcal{V}_{1234}^{o} := (2\pi)^4\left(1+e^{3k_1^0/\kappa}\right)\delta(k_4^0-k_3^0+k_2^0-k_1^0)\delta^{(3)}\left((\vec{k}_4-\vec{k}_3)e^{k_4^0/\kappa}+(\vec{k}_2-\vec{k}_1)e^{k_1^0/\kappa}\right),$$

and for the non-orientable model

$$\mathcal{V}_{1234}^{no} := (2\pi)^4\left(1+e^{-3(k_1^0+k_3^0)/\kappa}\right)\delta(k_4^0-k_3^0+k_2^0-k_1^0)\delta^{(3)}\left(\vec{k}_4-\vec{k}_3+\vec{k}_2e^{-k_4^0/\kappa}-\vec{k}_1e^{-k_1^0/\kappa}\right),$$

where we have defined $\mathcal{V}_{abcd}=\mathcal{V}(k_a,k_b,k_c,k_d)$.

<u>Remarks</u>: → Non-linearity of the spatial delta function;
→ The two models lead to very different QFT (UV/IR mixing…)

Path integral quantisation

- The various radiative corrections to the n-point functions can be obtained as usual by expanding the generating functional of connected correlation functions W up to the desired order in the coupling constant g.

→ This is a mere consequence of the representation of the NCFT as ordinary QFT.

- Explicitly,

$$W[J,\bar{J}] \coloneqq \ln Z[J,\bar{J}\,] = \mathrm{N} + \mathrm{W_G}[J,\bar{J}\,] + \ln\left(1 + e^{-W_G}\left(e^{-S_\kappa^{int}\left[\frac{\delta}{\delta \bar{J}},\frac{\delta}{\delta J}\right]} - 1\right)e^{W_G}\right),$$

$$\text{with } W_G[J,\bar{J}] \coloneqq \int \frac{d^4k}{(2\pi)^4}\bar{J}(k)\Delta(k)J(k).$$

The effective action Γ is then defined as the Legendre transform of W

$$\Gamma[\phi,\bar{\phi}] \coloneqq \int \frac{d^4k}{(2\pi)^4}\left(\bar{J}(k)\phi(k) + J(k)\bar{\phi}(k)\right) - W[J,\bar{J}],$$

$$\bar{\phi}(k) = \frac{\delta W}{\delta J(k)},\ \ \phi(k) = \frac{\delta W}{\delta \bar{J}(k)}.$$

One-loop 2-point function

- The quadratic part of the effective action, at one-loop order, is given by

$$\Gamma_1^{(2)}[\phi,\bar\phi] = \frac{1}{2}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\,\bar\phi(k_1)\phi(k_2)\Gamma_1^{(2)}(k_1,k_2),$$

$$\Gamma_1^{(2)}(k_1,k_2) := \frac{g}{(2\pi)^4}\int \frac{d^4k_3}{(2\pi)^4}\,\Delta(k)(\mathcal{V}_{3312}+\mathcal{V}_{1233}+\mathcal{V}_{1332}+\mathcal{V}_{3213})\,.$$

- Orientable model:

(all the $\mathcal{V}_{abcd}$ reduces to $\delta^{(4)}(k_2-k_1)$ times some powers of the twist factor)

$$\Gamma_1^{(2)}(k_1,k_2) = \int \frac{d^4k}{(2\pi)^4}\,\bar\phi(k)\big(\omega_1+\omega_2 e^{-3k^0/\kappa}\big)\phi(k),$$

$\Rightarrow$ The tree-level structure of the 2-point function is preserved.

- <u>Non-orientable model</u>:
  (not all the $\mathcal{V}_{abcd}$ reduces to $\delta^{(4)}(k_2 - k_1) \to$ non-planar contributions)

  (i) The planar contributions are

$$\Gamma^{(2)}_{1,P}(k_1, k_2) = \int \frac{d^4k}{(2\pi)^4} \bar{\phi}(k)\left(\omega_3 + \omega_4 e^{-3k^0/\kappa}\right)\phi(k),$$

  $\Rightarrow$ Again, the tree-level structure of the 2-point function is preserved.
  Actually, $\omega_3 = \omega_2$ and $\omega_4 = \omega_1$.

  (ii) The non-planar contributions are

$$\Gamma^{(2)}_{1,NP}(k_1, k_2) = \delta(k_2^0 - k_1^0)\left(\Xi(k_1, k_2) + \Xi(k_2, k_1)\right),$$

$$\Xi(k_a, k_b) \coloneqq \int \frac{d^4k}{(2\pi)^4}\left(1 + e^{-3(k^0 + k_a^0)/\kappa}\right)\Delta(k)\delta^{(3)}\left(\left(1 - e^{-k_a^0/\kappa}\right)\vec{k} - \vec{k}_a + e^{-k^0/\kappa}\vec{k}_b\right).$$

  $\to$ One can easily show that $|\Xi(k_a, k_b)| < \infty$ at non-exceptional external momenta,

  while $$\Gamma^{(2)}_{1,NP}(k_1, 0) = \Gamma^{(2)}_{1,P}(k_1, 0),$$

  which diverges (see below) $\Rightarrow$ indicating UV/IR mixing.

$\rightarrow$ It is sufficient to compute ω_1 and ω_2.

- <u>Model with Casimir kinetic operator</u>:

$$\omega_j = \frac{g\kappa}{\pi}\int_0^\infty dy\, \Phi_j(y) I(y) \text{ with } \Phi_{1(y)} := \frac{2}{1+y^3} + 1 \text{ and } \Phi_2(y) := \frac{2}{1+y^3} + \frac{1}{y^3},$$

$$I(y) := \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\|\vec{k}\|^2 + \kappa^2 \mu_c^2(y)}, \ \mu_c^2(y) := 1 - \frac{2\kappa^2 - m^2}{\kappa^2} y + y^2.$$

$\rightarrow$ "ordinary" space-like integral times some function of y (~time-like variable)

- Regularisation scheme:

(a) Pauli-Villars regularisation to extract the ("spatial") singular behaviour of I

$$I(y) \rightarrow I_\Lambda(y) := \mathrm{I(y)} - \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \left(\|\vec{k}\|^2 + \Lambda^2\right)^{-1}$$

$$\frac{1}{A^a B^b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 du\, \frac{u^{a-1}(1-u)^{b-1}}{(uA + (1-u)B)^{a+b}}, \qquad a, b > 0,$$

$$\int \frac{d^n p}{(2\pi)^n} (p^2 + M^2)^{-m} = M^{n-2m} \frac{\Gamma(m - n/2)}{(4\pi)^{n/2}\Gamma(m)}, \qquad m > \frac{n}{2} > 0.$$

$$\Rightarrow \ \omega_j(\Lambda, \Lambda_0) = \frac{\kappa}{(2\pi)^2} \int_{\Lambda_0} dy\, \Phi_j(y) \big(\Lambda - \kappa \mu_c(y)\big).$$

(b) Cut-off regularisation of the y-integral

Why y variable? By definition $y := e^{-k^0/\kappa}$. We recognize $\mathcal{E} := e^{-P_0/\kappa}$, a generator of the κ-deformed translation Hopf sub-algebra. Also,

$$C_\kappa \sim \|\vec{\mathcal{P}}\|^2 + \mathcal{P}_0^2, \text{ with } \vec{\mathcal{P}} := \vec{k} \text{ and } \mathcal{P}_0 := \kappa(1-y) \xrightarrow[\kappa\to\infty]{} k^0.$$

$\Rightarrow \mathcal{P}_0$ seems to be the natural quantity replacing the energy within this framework.

For these reasons, we regularise the time-like integral by cutting $\mathcal{P}_0$ instead of the Fourier parameter k^0, namely $|\mathcal{P}_0| \leq \Lambda_0$. This implies

$$\frac{\kappa}{\kappa+\Lambda_0} \leq y \leq \frac{\kappa+\Lambda_0}{\kappa},$$

$$\text{s.t. } \int_0^\infty dy \to \int_{\Lambda_0} dy \equiv \int_{\frac{\kappa}{\kappa+\Lambda_0}}^{\frac{\kappa+\Lambda_0}{\kappa}} dy\,.$$

- Standard computations finally yield ($\Lambda = \Lambda_0$)

$$\omega_1(\Lambda) = \frac{\Lambda^2}{8\pi^2} + c_1(\kappa)\kappa\Lambda - \mathrm{c}_2(\kappa)\, m^2 \ln\frac{\Lambda}{\kappa} + F_1(\kappa),$$
$$\omega_2(\Lambda) = \frac{\Lambda^3}{8\pi^2\kappa} + \frac{\Lambda^2}{8\pi^2} + c_1'(\kappa)\kappa\Lambda - \mathrm{c}_2'(\kappa)\, m^2 \ln\frac{\Lambda}{\kappa} + F_2(\kappa),$$

where $F_j(\kappa)$ are finite terms and the coefficients are dimensionless.

- Model with equivariant kinetic operator:

$$\omega_j = \frac{g\kappa}{\pi} \int_0^\infty dy \, \Phi_j(y) I(y)$$

$$I(y) := \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\left(\|\vec{k}\|^2 + \kappa^2 \mu_+^2(y)\right)\left(\|\vec{k}\|^2 + \kappa^2 \mu_-^2(y)\right)}, \quad \mu_\pm^2(y) := 1 \pm 2\sqrt{1 - \frac{m^2}{\kappa^2}}\, y + y^2.$$

→ The space-like integral is finite.

$$\Rightarrow \omega_j(\Lambda_0) = \frac{\kappa^3}{4\pi^2\sqrt{\kappa^2 - m^2}} \int_{\Lambda_0} dy \, \Phi_j(y)\big(\mu_+(y) - \mu_-(y)\big),$$

$$\Rightarrow \omega_j(\Lambda_0) = \frac{\kappa\Lambda_0}{2\pi^2} + F_j(\kappa).$$

→ Milder quantum behaviour than in the commutative case.

One-loop 4-point function

- We now focus on the model with equivariant kinetic operator and orientable interactions.

- The one-loop quartic part of the effective action is given by

$$\Gamma_1^{(4)}[\phi,\bar\phi] = \tfrac{1}{4!}\int \prod_{\ell=1}^{4} \frac{d^4k_\ell}{(2\pi)^4}\,\bar\phi(k_1)\phi(k_2)\bar\phi(k_3)\phi(k_4)\Gamma_1^{(4)}(k_1,k_2,k_3,k_4),$$

with $\Gamma_1^{(4)}(k_1,k_2,k_3,k_4) = \frac{g^2}{(2\pi)^8}\int \frac{d^4k_5}{(2\pi)^4}\frac{d^4k_6}{(2\pi)^4}\Delta(k_5)\Delta(k_6)\,\times$

$$\begin{aligned}\times\big(&2\mathcal{V}_{5462}\mathcal{V}_{3615} + 2\mathcal{V}_{5462}\mathcal{V}_{3516} + 2\mathcal{V}_{5216}\mathcal{V}_{3465} + \\ &+2\mathcal{V}_{1652}\mathcal{V}_{3465} + 2\mathcal{V}_{5612}\mathcal{V}_{6435} + 2\mathcal{V}_{5612}\mathcal{V}_{3564} + \\ &+2\mathcal{V}_{5612}\mathcal{V}_{3465} + \mathcal{V}_{5612}\mathcal{V}_{6534} + \mathcal{V}_{1256}\mathcal{V}_{3465} + \\ &+2\mathcal{V}_{5216}\mathcal{V}_{3564} + \mathcal{V}_{5216}\mathcal{V}_{6435} + \mathcal{V}_{1652}\mathcal{V}_{3564}\big).\end{aligned}$$

→ The non-planar contributions are finite with no singularities in the external momenta.

- Making use of the symmetry properties

$$V_{1234} = V_{4321}, \qquad V_{1234} = e^{3\left(k_3^0 - k_4^0\right)/\kappa} V_{3412},$$

together with the fusion rules

$$V_{12\mathbf{65}} V_{\mathbf{56}34} = V_{\mathbf{56}34} \mathrm{V}_{1234}, \qquad \mathrm{V}_{1\mathbf{56}4} V_{\mathbf{5}23\mathbf{6}} = V_{\mathbf{5}23\mathbf{6}} V_{1234},$$

one can gather the various contributions s.t.

$$\Gamma_{1,a}^{(4)}(k_1,k_2,k_3,k_4) = \frac{g^2}{(2\pi)^8} \int \frac{d^4k_5}{(2\pi)^4} \frac{d^4k_6}{(2\pi)^4} \Delta(k_5)\Delta(k_6)\Psi_1\left(k_5^0\right) V_{\mathbf{56}34} \mathrm{V}_{1234},$$

$$\Gamma_{1,b}^{(4)}(k_1,k_2,k_3,k_4) = \frac{g^2}{(2\pi)^8} \int \frac{d^4k_5}{(2\pi)^4} \frac{d^4k_6}{(2\pi)^4} \Delta(k_5)\Delta(k_6)\Psi_1\left(k_5^0\right) V_{\mathbf{5}23\mathbf{6}} \mathrm{V}_{1234}.$$

- Familly using the following estimate on the equivariant propagator

$$\Delta\left(y,\vec{k}\right) \leq \frac{8\kappa^2 y^2}{1+y^3} \left(\left\|\vec{k}\right\|^2 + \kappa^2 \mu_-^2(y) \right)^{-2},$$

one can easily infer a bound on $\Gamma_{1,j}^{(4)}$. One finds $\left|\Gamma_{1,j}^{(4)}\right| < \infty, j = a, b$.

Summary & outlook

- We have computed the one-loop order corrections to both the 2-point and 4-point functions for various (4) models of κ-Poincaré invariant complex scalar field theory with quartic interactions. → New results in NCQFT. Explore in more details their implications in HEP, QG…
- The computations appear to be very simple and the integral expression for the star product enables us to make use of the standard machinery from (ordinary) QFT. → encouraging for undertaking further studies in NCFT/GT.
- We show that the models with *Casimir kinetic operator* require to consider the 4 interactions to ensure the two component of the mass-operator to be renormalised the same way & One-loop 2-point function *diverge cubically* with UV/IR mixing.
- On the other hand, the models with *equivariant kinetic operator diverge linearly* with possibly UV/IR mixing. The 4-point function is finite. Hence, better quantum behaviour for model with orientable interactions than in the commutative case. → Probably renormalisable at all order (to be proved).

- Both kinetic operators are IR singular for massless theory. → Study massless theories.

- If one want to renormalized the theory which has UV/IR mixing, one should first extract the (IR) singularity from the non-planar contributions.

Thank you.