# The non-perturbative physics of membrane matrix models—a phase diagram for the BMN model

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Y. Asano, V. Filev, S. Kováčik and D.O'C [arXiv:1805.05314]
V. Filev and D.O'C. [arXiv:1506.01366 and 1512.02536]

- Introduction
- From Membranes to Matrices
- The BFSS model,
- Membranes on other backgrounds
- The BMN model
- The phase diagram
- Where to go from here.

# Membrane Actions

## Nambu Goto—the simplest: On p-brane

$S_{NG} = \int_{\mathcal{M}} d^{p+1}x\sqrt{-detG} \qquad G_{\mu\nu} = \partial_\mu X^M \partial_\nu X^N g_{MN}(X)$

## Higher form gauge field on the world volume

$$S_{p-form} = -\int_{\mathcal{M}} \frac{1}{(p+1)!}\epsilon^{\mu_1\ldots\mu_{p+1}} C_{\mu_1\ldots\mu_{p+1}}$$

$$C_{\mu_1\ldots\mu_{p+1}} = \partial_{\mu_1} X^{M_1} \ldots \partial_{\mu_{p+1}} X^{M_{p+1}} C_{M_1\ldots M_{p+1}}$$

We could add

- an anti-symmetric part to $G_{\mu\nu}$ to get a Dirac-Born-Infeld action.
- extrinsic curvature terms.

Supersymmetric $S_{NG}$ exist only in 4, 5, 7 and 11 dim-spacetime.

# Membrane sigma models

**The Membrane action, Polyakov form – sigma model**

$$S_{NG} = -\frac{T}{2}\int_{\mathcal{M}} d^3\sigma\sqrt{-h}\left(h^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N g_{MN} - \Lambda\right)$$

Choose $\Lambda = 1$ (rescale $X^a$ and $T$).

**Eliminating $h_{\mu\nu}$**

$$h_{\alpha\beta} = \partial_\alpha X^M\partial_\beta X^N g_{MN} = G_{\alpha\beta}$$

returns us to Nambu-Goto.

For p-branes set $\Lambda = p - 1$.

# Membranes in flat spacetime, $g_{MN} = \eta_{MN}$ and $C_3 = 0$

For membrane topology $\mathbb{R} \times \Sigma$ we can set $h_{0i} = 0$ and $h_{00} = -\frac{4}{\rho}\mathrm{det}(h_{ij})$.

The action becomes

$$S = \frac{T\rho}{4} \int dt \int_\Sigma d^2\sigma \left( \dot{X}^M \dot{X}^N \eta_{MN} - \frac{4}{\rho^2}\mathrm{det}(h_{ij}) \right)$$

In 2-dim $det(h_{ij})$ can be rewritten using $\{f,g\} = \epsilon^{ij}\partial_i f \partial_j g$ as

$$S = \frac{T\rho}{4}\int dt \int_\Sigma d^2\sigma \left(\dot{X}^M \dot{X}^N \eta_{MN} - \frac{4}{\rho^2}\{X^M, X^N\}^2\right)$$

and the constraints become

$$\dot{X}^M \partial_i X_M = 0 \implies \{\dot{X}^M, X_M\} = 0$$
$$\text{and} \qquad \dot{X}^M \dot{X}_M = -\frac{2}{\rho^2}\{X^M, X^N\}\{X_M, X_N\}\,.$$

Using lightcone coordinates with $X^\pm = (X^0 \pm X^{D-1})/\sqrt{2}$ with $X^+ = \tau$ we can solve the constraint for $\dot{X}^-$ and Legendre transform to the Hamiltonian to find

$$S = -T\int \sqrt{-G} \longrightarrow H = \int_\Sigma \left(\frac{1}{\rho T} P^a P^a + \frac{T}{2\rho}\{X^a, X^b\}^2\right)$$

With the remaining constraint $\{P^a, X^a\} = 0$.

Noting for higher p-branes the procedure works the same and using

$$\det(\partial_i X^a \partial_j X^b h_{ab}) =$$
$$\frac{1}{p!}\{X^{a_1}, X^{a_2} \ldots, X^{a_p}\}\{X^{b_1}, X^{b_2} \ldots, X^{b_p}\} h_{a_1 b_1} h_{a_2 b_2} \ldots h_{a_p b_p}$$
$$\{X^{a_1}, X^{a_2} \ldots, X^{a_p}\} := \epsilon^{j_1, j_2, \ldots, j_p} \partial_{j_1} X^{a_1} \partial_{j_2} X^{a_2} \ldots \partial_{j_p} X^{a_p}$$

and the Hamiltonian becomes

$$H = \int_\Sigma d^p\sigma \left( \frac{1}{\rho T} P^a P^a + \frac{4}{p!\rho^2} \{X^{a_1}, X^{a_2} \ldots, X^{a_p}\}^2 \right)$$

# Quantisation

A direct apporach, either Hamiltonian or path integral, has not yet been successful.

**Matrix membranes**

Functions are approximated by $N \times N$ matrices, $f \to \mathrm{F}$, and $\int_\Sigma f \to \mathrm{TrF}$.

The Hamiltonian becomes

$$\mathrm{H} = -\frac{1}{2}\nabla^2 - \frac{1}{4}\sum_{a,b=1}^{D} \mathrm{Tr}[X^a, X^b]^2$$

restricted to $U(N)$ singlet "physical" states.

- $H$ describes a "fuzzy" membrane in $D+1$ spacetime.
- Much of the classical topology and geometry are lost.
- At low energy, or the bottom of the potential $[X^a, X^b] = 0$.

Once we have the Hamiltonian $H$ we can consider thermal ensembles of membranes whose partition function is given by

$$Z = \mathrm{Tr}_{Phys}(\mathrm{e}^{-\beta H})$$

where the physical constraint means the states are $U(N)$ invariant.

## Path Integral version

$$Z = \int [dX] \mathrm{e}^{-\int_0^\beta d\tau \, \mathrm{Tr}(\frac{1}{2}(D_\tau X^a) - \frac{1}{4}[X^a, X^b]^2)}$$

## Gauss law constraint

The projection onto physical states — the Gauss law constraint is implemented by the gauge field with

$$D_\tau X^a = \partial_\tau X^a - i[A, X^a].$$

Matrix membrane models are the zero volume limit of Yang-Mills compactified on a torus.

# Understanding gauged quantum matrix models

The simplest example of a quantum mechanical model with Gauss Law constraint is a set of $p$ gauged Gaussians. Their Euclidean actions are

$$N\int_0^\beta \mathrm{Tr}(\frac{1}{2}(\mathcal{D}_\tau X^i)^2+\frac{1}{2}m^2(X^i)^2)$$

$\mathcal{D}_\tau X^i = \partial_\tau X^i - i[A, X^i]$.

# Properties of gauge gaussian models

- The eigenvalues of $X^i$ have a Wigner semi-circle distribution.
- At $T = 0$, we can gauged $A$ away, while for large $T$ we get a pure matrix model with $A$ one of the matrices.
- The entry of $A$ as an additional matrix in the dynamics signals a phase transition. In the Gaussian case with $p$ scalars it occurs at

  $$T_c = \frac{m}{\ln p}$$

  The transition can be observed as centre symmetry breaking in the Polyakov loop.

Bosonic matrix membranes are approximately gauge gaussian models V. Filev and D.O'C. [1506.01366 and 1512.02536]. They have however two phase transitions, very close in temperature.

# The BFSS model

$$S_{SMembrane} = \int \sqrt{-G} - \int C + \quad \text{Fermionic terms}$$

The susy version only exists in $4, 5, 7$ and $11$ spacetime dimensions.

**BFSS Model — The supersymmetric membrane à la Hoppe**

$$\mathrm{H} = \mathrm{Tr}\left(\frac{1}{2}\sum_{a=1}^{9} P^a P^a - \frac{1}{4}\sum_{a,b=1}^{9}[X^a, X^b][X^a, X^b] + \frac{1}{2}\Theta^T\gamma^a[X^a, \Theta]\right)$$

The model is claimed to be a non-perturbative 2nd quantised formulation of $M$-theory.

A system of N interacting D0 branes.

Note the flat directions.

# Finite Temperature Model

The partition function and Energy of the model at finite temperature is

$$Z = Tr_{Phys}(\mathrm{e}^{-\beta\mathcal{H}}) \qquad \text{and} \qquad E = \frac{Tr_{Phys}(\mathcal{H}\mathrm{e}^{-\beta\mathcal{H}})}{Z} = \langle\mathcal{H}\rangle$$

The 16 fermionic matrices $\Theta_\alpha = \Theta_{\alpha A} t^A$ are quantised as

$$\{\Theta_{\alpha A}, \Theta_{\beta B}\} = 2\delta_{\alpha\beta}\delta_{AB}$$

The $\Theta_{\alpha A}$ are $2^{8(N^2-1)}$ and the Fermionic Hilbert space is

$$\mathcal{H}^F = \mathcal{H}_{256} \otimes \cdots \otimes \mathcal{H}_{256}$$

with $\mathcal{H}_{256} = \mathbf{44} \oplus \mathbf{84} \oplus \mathbf{128}$ suggestive of
the graviton (**44**), anti-symmetric tensor (**84**) and gravitino (**128**)
of $11 - d$ SUGRA.

For an attempt to find the ground state see: J. Hoppe et al
arXiv:0809.5270

# Lagrangian formulation

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The BFSS matrix model is also the dimensional reduction of ten dimensional supersymmetric Yang-Mills theory down to one dimension:

$$\begin{aligned} S_{BFSS} &= \int d\tau \, \mathrm{Tr} \left\{ \frac{1}{2}(\mathcal{D}_\tau X^i)^2 - \frac{1}{4}[X^i, X^j]^2 \right. \\ & \left. + \frac{1}{2}\Psi^T D_\tau \Psi + \frac{1}{2}\Psi^T \Gamma^i [X^i, \Psi] \right\} \ , \end{aligned}$$

where $\Psi$ is a thirty two component Majorana–Weyl spinor, $\Gamma^i$ are gamma matrices of $Spin(9)$.

# The gravity dual and its geometry

Gauge/gravity duality predicts that the strong coupling regime of the theory is described by $II_A$ supergravity, which lifts to 11-dimensional supergravity.

The bosonic action for eleven-dimensional supergravity is given by

$$S_{11D} = \frac{1}{2\kappa_{11}^2}\int[\sqrt{-g}R - \frac{1}{2}F_4 \wedge *F_4 - \frac{1}{6}A_3 \wedge F_4 \wedge F_4]$$

where $2\kappa_{11}^2 = 16\pi G_N^{11} = \frac{(2\pi l_p)^9}{2\pi}$.

The relevant solution to eleven dimensional supergravity for the dual geometry to the BFSS model corresponds to $N$ coincident $D0$ branes in the IIA theory. It is given by

$$ds^2 = -H^{-1}dt^2 + dr^2 + r^2 d\Omega_8^2 + H(dx_{10} - Cdt)^2$$

with $A_3 = 0$
The one-form is given by $C = H^{-1} - 1$ and $H = 1 + \frac{\alpha_0 N}{r^7}$ where $\alpha_0 = (2\pi)^2 14\pi g_s l_s^7$.

# Including temperature

The idea is to include a **black hole** in the gravitational system.

The Hawking termperature provides the temperature of the system.

### Hawking radiation

We expect difficulties at low temperatures, as the system should Hawking radiate. It is argued that this is related to the flat directions and the propensity of the system to leak into these regions.

# The black hole geometry

$$ds_{11}^2 = -H^{-1}Fdt^2 + F^{-1}dr^2 + r^2 d\Omega_8^2 + H(dx_{10} - Cdt)^2$$

Set $U = r/\alpha'$ and we are interested in $\alpha' \to \infty$
$H(U) = \frac{240\pi^5\lambda}{U^7}$ and the black hole time dilation factor
$F(U) = 1 - \frac{U_0^7}{U^7}$ with $U_0 = 240\pi^5\alpha'^5\lambda$. The temperature

$$\frac{T}{\lambda^{1/3}} = \frac{1}{4\pi\lambda^{1/3}} H^{-1/2}F'(U_0) = \frac{7}{2^4 15^{1/2}\pi^{7/2}}\left(\frac{U_0}{\lambda^{1/3}}\right)^{5/2}.$$

**From black hole entropy we obtain the prediction for the Energy**

$$S = \frac{A}{4G_N} \sim \left(\frac{T}{\lambda^{1/3}}\right)^{9/2} \implies \frac{E}{\lambda N^2} \sim \left(\frac{T}{\lambda^{1/3}}\right)^{14/5}$$

# Checks of the predictions

We found excellent agreement with this prediction V. Filev and D.O'C. [1506.01366 and 1512.02536].
The best current results (Berkowitz et al 2016) consistent with gauge gravity give

$$\frac{1}{N^2}\frac{E}{\lambda^{1/3}} = 7.41\left(\frac{T}{\lambda^{1/3}}\right)^{\frac{14}{5}} - (10.0 \pm 0.4)\left(\frac{T}{\lambda^{1/3}}\right)^{\frac{23}{5}} + (5.8 \pm 0.5)T^{\frac{29}{5}} + \dots - \frac{5.77T^{\frac{2}{5}} + (3.5 \pm 2.0)T^{\frac{11}{5}}}{N^2} + \dots$$

# Membranes on other backgrounds

There are many options for background geometries:

**PP-Wave backgrounds**

Two options that lead to massive deformations of the BFSS model

**N=1***

Breaks susy down to 4 remaining.

**BMN model**

Preserves all 16 susys and has $SU(4|2)$ symmetry.

# The BMN or PWMM

The supermembrane on the maximally supersymmetric plane wave spacetime

$$ds^2 = -2dx^+dx^- + dx^a dx^a + dx^i dx^i - dx^+ dx^+ ((\frac{\mu}{6})^2 (x^i)^2 + (\frac{\mu}{3})^2 (x^a)^2)$$

with

$$dC = \mu dx^1 \wedge dx^2 \wedge dX^3 \wedge dx^+$$

so that $F_{123+} = \mu$. This leads to the additional contribution to the Hamiltonian

$$\Delta \mathrm{H}_\mu = \frac{N}{2} \mathbf{Tr} \Big( (\frac{\mu}{6})^2 (X^a)^2 + (\frac{\mu}{3})^2 (X^i)^2 + \frac{2\mu}{3} i\epsilon_{ijk} X^i X^j X^k + \frac{\mu}{4} \Theta^T \gamma^{123} \Theta \Big)$$

# The BMN model

## The BMN action

$$
\begin{aligned}
S_{BMN} \;=\; \int_0^\beta d\tau \, \mathrm{Tr} \Big\{ & \frac{1}{2}(\mathcal{D}_\tau X^i)^2 + (\frac{\mu}{6})^2 (X^a)^2 + (\frac{\mu}{3})^2 (X^i)^2 \\
& + \Psi^T D_\tau \Psi + \frac{\mu}{4} \Psi^T i \gamma^{123} \Psi \\
& - \frac{1}{4}[X^i, X^j]^2 + \frac{2\mu}{3} i \epsilon_{ijk} X^i X^j X^k + \frac{1}{2} \Psi^T \Gamma^i [X^i, \Psi] \Big\} \; ,
\end{aligned}
$$

# Large mass expansion

For large $\mu$ the model becomes the supersymmetric Gaussian model

## Finite temperature Euclidean Action

$$S_{BMN} = \frac{1}{2g^2}\int_0^\beta d\tau\,\mathrm{Tr}\Big\{(\mathcal{D}_\tau X^i)^2 + (\frac{\mu}{6})^2(X^a)^2 + (\frac{\mu}{3})^2(X^i)^2 \\ \Psi^T D_\tau \Psi + \frac{\mu}{4}\Psi^T\gamma^{123}\Psi\Big\}$$

This model has a phase transition at $T_c = \frac{\mu}{12\ln 3}$

# Perturbative expansion in large $\mu$.

Three loop result of *Hadizadeh, Ramadanovic, Semenoff and Young* [hep-th/0409318]

$$T_c = \frac{\mu}{12\ln 3}\left\{1 + \frac{2^6\times 5}{3^4}\frac{\lambda}{\mu^3} - (\frac{23\times 19927}{2^2\times 3^7} + \frac{1765769\ln 3}{2^4\times 3^8})\frac{\lambda^2}{\mu^6} + \cdots\right\}$$

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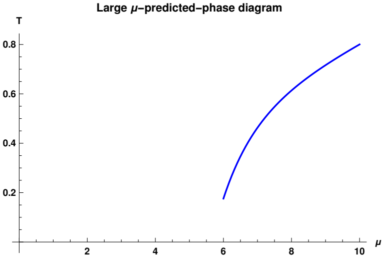


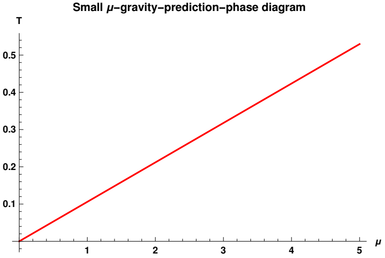

Passes through zero at $\mu = 13.4$.

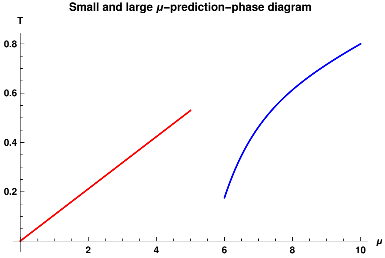
# Gravity prediction at small $\mu$

Costa, Greenspan, Penedones and Santos, [arXiv:1411.5541]

$$\lim_{\frac{\lambda}{\mu^2}\to\infty} \frac{T_c^{\mathrm{SUGRA}}}{\mu} = 0.105905(57)\,.$$

The prediction is for low temperatures and small $\mu$ the transition temperature approaches zero linearly in $\mu$.



# Padé approximant prediction of $T_c$

$$T_c = \frac{\mu}{12\ln 3}\left\{1 + r_1\frac{\lambda}{\mu^3} + r_2\frac{\lambda^2}{\mu^6} + \cdots\right\}$$

with $r_1 = \frac{2^6\times 5}{3}$ and $r_2 = -\left(\frac{23\times 19927}{2^2\times 3} + \frac{1765769\ln 3}{2^4\times 3^2}\right)$

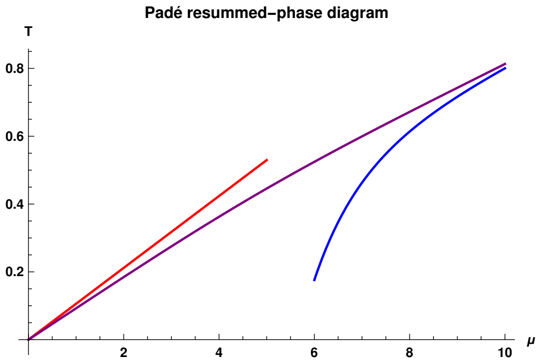
Using a Padé Approximant: $1 + r_1 g + r_2 g^2 + \cdots \to 1 + \frac{1+r_1 g}{1-\frac{r_2}{r_1}g}$

$$\Longrightarrow\ T_c^{\text{Padé}} = \frac{\mu}{12\ln 3}\left\{1 + \frac{r_1\frac{\lambda}{\mu^3}}{1 - \frac{r_2}{r_1}\frac{\lambda}{\mu^3}}\right\}$$

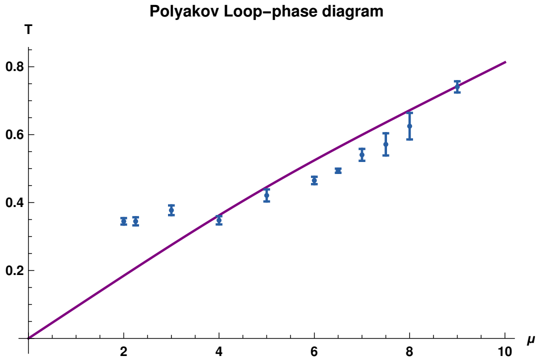
Now we can take the small $\mu$ limit

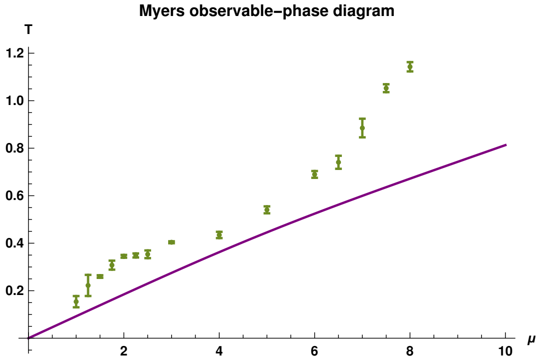
$$\lim_{\frac{\lambda}{\mu^2}\to\infty} \frac{T_c^{\text{Padé}}}{\mu} \simeq \frac{1}{12\ln 3}\left(1-\frac{r_1^2}{r_2}\right) = 0.0925579$$

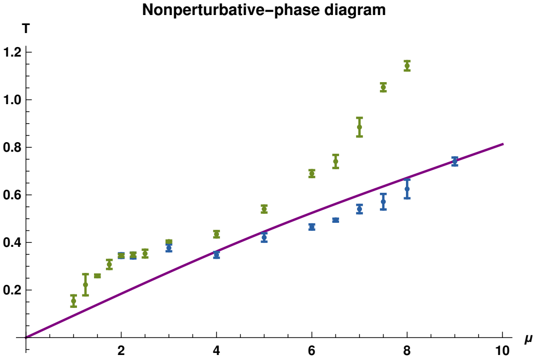
$$\lim_{\frac{\lambda}{\mu^2}\to\infty} \frac{T_c^{\text{SUGRA}}}{\mu} = 0.105905(57)\,.$$



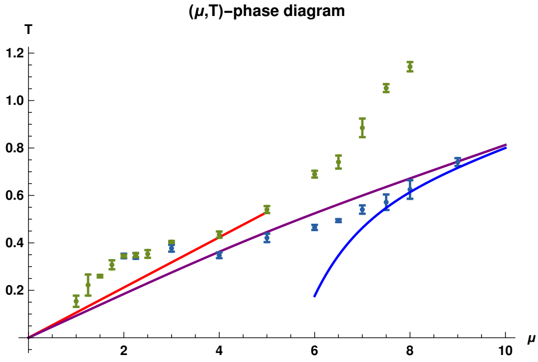
# A non-perturbative phase diagram from the Polyalov Loop.




Myers observable-phase diagram
T
1.2
1.0
0.8
0.6
0.4
0.2
2
4
6
8
10
μ




**Green** Myers transition
**Blue** Polyakov loop transition
**Purple** Padé prediction for the transition


(μ,T)–phase diagram
T
1.2
1.0
0.8
0.6
0.4
0.2
2
4
6
8
10
μ

# 4-parameter Lattice discretisation

## The bosonic lattice Laplacian

$$\Delta_{Bose} = \Delta + r_b a^2 \Delta^2 \,, \quad \text{where} \quad \Delta = \frac{2 - e^{aD_\tau} - e^{-aD_\tau}}{a^2} \,.$$

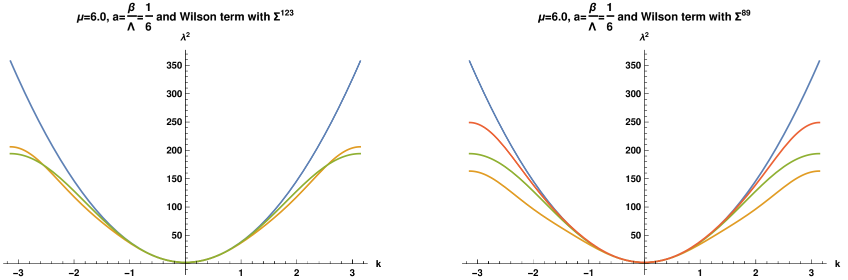
## Lattice Dirac operator

$$D_{Lat} = K_a \mathbf{1}_{16} - i\frac{\mu}{4}\gamma^{567} + \Sigma^{123} K_w \,, \quad \text{where} \quad \Sigma^{123} = i\gamma^{123} \,.$$

$$K_a = (1-r)\frac{e^{aD_\tau} - e^{-aD_\tau}}{2a} + r\frac{e^{2aD_\tau} - e^{-2aD_\tau}}{4a} \qquad \text{lattice derivative}$$

$$K_w = r_{1f}\, a\Delta + r_{2f}\, a^3 \Delta^2 \qquad \text{the Wilson term}$$

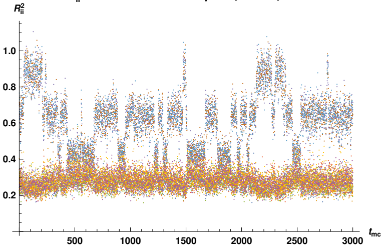
# Lattice Dispersion relations



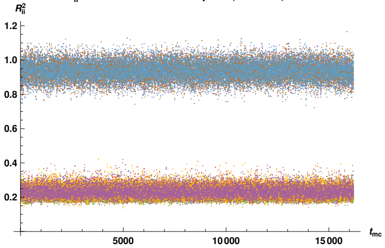

Eigenvalues $k^2 + \frac{\mu^2}{4}$ (blue parabola),
$\Delta_{Bose} + \frac{\mu^2}{4}$ light green,
$\Sigma^{89}$ splitting red and orange curves.

# Observables

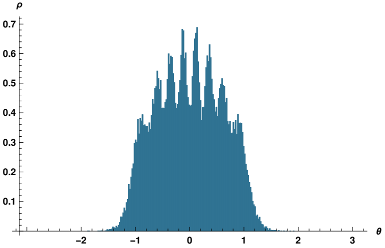
$R_{ii}^2$ vs simulation time for $\mu$=6.0, T=0.69, N=8



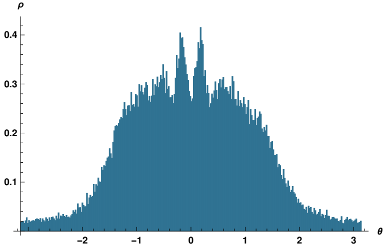

$R_{ii}^2$ vs simulation time for $\mu$=6.0, T=0.365, N=8

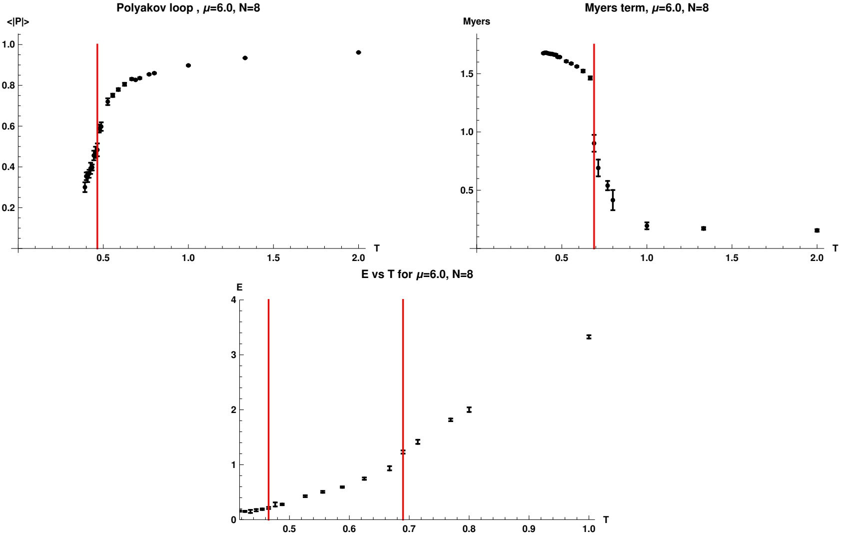



Eigenvalue density of A for $\mu$=6.0, T=0.69, N=8, Λ=24

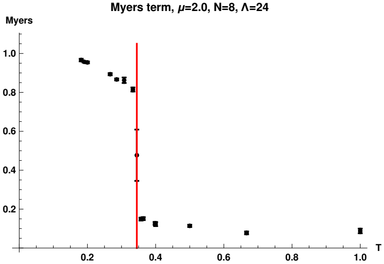


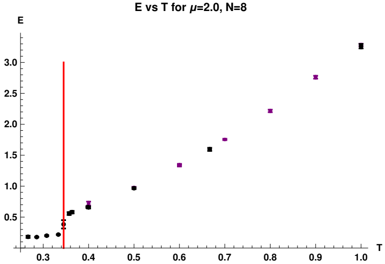

Eigenvalue density of A for $\mu$=6.0, T=0.465, N=8, Λ=24

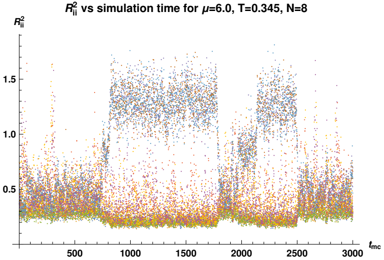


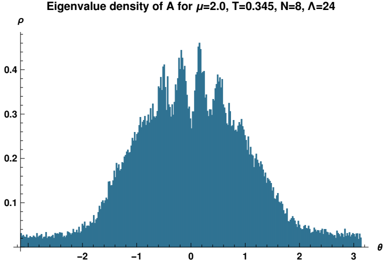

Polyakov loop , μ=6.0, N=8
<|P|>
T
Myers term, μ=6.0, N=8
Myers
T
E vs T for μ=6.0, N=8
E
T

# Small $\mu$

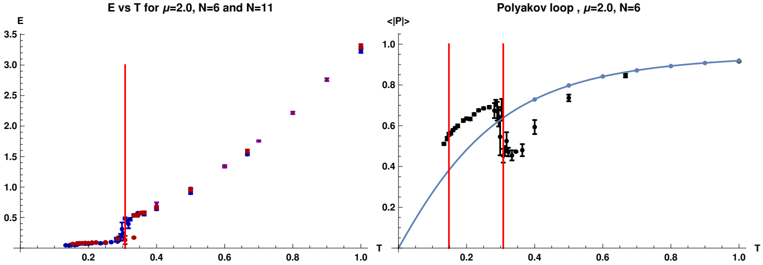

Myers term, μ=2.0, N=8, Λ=24
Myers
1.0
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1.0
T


E vs T for μ=2.0, N=8
E
3.0
2.5
2.0
1.5
1.0
0.5
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
T


R²ᵢᵢ vs simulation time for μ=6.0, T=0.345, N=8
R²ᵢᵢ
1.5
1.0
0.5
500
1000
1500
2000
2500
3000
t_mc


Eigenvalue density of A for μ=2.0, T=0.345, N=8, Λ=24
ρ
0.4
0.3
0.2
0.1
−2
−1
0
1
2
3
θ

# Non-monotonic Polyakov loop



# Where do we go from here

- Study the bosonic BMN model—its phase diagram, theoretical predictions.
- Implications of $SU(4|2)$ symmetry.
- M2-branes.
- Probe BMN with D4-branes—already coded.
- $N = 1^*$ model — at coding stage.
- $N = 2$ models.
- Black dual geometries?
- M5-brane matrix models?

# Conclusions

- Bosonic membranes quantised a la Hoppe are well approximated as massive gauged gaussian models.
- Tests of the BFSS model against non-perturbative studies are in excellent agreement.
- It is useful to have probes of the geometry.
- The mass deformed model, i.e. the BMN model is more complicated. Initial phase diagrams indicate agreement with gravity predictions
- But ...
- More work is needed. A study of non-spherical type IIA black holes would be very useful.

Thank you for your attention!