Integrability and Generalized Monodromy Matrix of Two Dimensional String Effective Action

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Plan de la présentation

Sommaire

Introduction

- Two-Dimensional field theories play an important role in describing a variety of physical systems
- These Models possess the interesting property of integrability
- Two-Dimensional non linear sigma models are endowed with a rich symmetry structure of string theories
- T-Duality Symmetries have played a very important role in the understanding of string dynamics
- The tree-level string effective action compactified on a d-dimensional torus T^d is known to be invariant under the non compact global symmetry group $O(d, d)$

Introduction

- The construction of the Monodromy Matrix turns out to be one of the principle objectives in the study of integrable systems
- The integrability of dimensionally reduced gravity and supergravity to two dimensions has been studied extensively by introducing the spectral parameter and currents under a local $O(d) \times O(d)$ transformation
- Construction of generalized Monodromy Matrix $\widehat{M}(\omega)$ from general Integrability conditions in terms of general functions under T-duality symmetry

Sommaire

Two Dimensional String Effective Action

We are interested in the two-dimensional σ model coupled to gravity, namely

$$S = \int dx^2 \sqrt{-g} e^{-\overline{\Phi}} [R + (\partial \overline{\Phi})^2 + \frac{1}{8} Tr(\partial_\alpha M^{-1} \partial^\alpha M)] \qquad (1)$$

The field $\overline{\Phi}$ is the usual shifted dilaton given by

$$\overline{\Phi} = \Phi - \frac{1}{2}\, log\ det G_{ij} \qquad (2)$$

with $i, j = 2, 3, ..., (D-1), d = D-2$ and
M *is a* $2d \times 2d$ symmetric matrix

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \qquad (3)$$

Two Dimensional String Effective Action

G and B parametrize the coset $O(d,d)/O(d) \times O(d)$
The action (1) is invariant under global $O(d,d)$ transformations, namely

$$g_{\alpha\beta} \to g_{\alpha\beta} \tag{4}$$

$$\overline{\Phi} \to \overline{\Phi}$$

$$M \to \Omega^T M \Omega \ , \ \ \Omega \in O(d,d)$$

and the variation with respect to M leads to the conservation law

$$\partial_\alpha [e^{-\overline{\Phi}} \sqrt{-g} g^{\alpha\beta} M^{-1} \partial_\beta M] = 0 \tag{5}$$

Two Dimensional String Effective Action

In order to make apparent a local $O(d) \times O(d)$ as well as a global $O(d,d)$ transformation, it is convenient to introduce a triangular matrix V contained in $O(d,d)/O(d) \times O(d)$ of the following form

$$V = \begin{pmatrix} E^{-1} & 0 \\ BE^{-1} & E^T \end{pmatrix} \tag{6}$$

such that

$$M = VV^T \ , \ (E^T E)_{ij} = G_{ij}$$

$$V \to \Omega^T V h(x) \tag{7}$$

where $\Omega \in O(d,d)$ and $h(x) \in O(d) \times O(d)$

Two Dimensional String Effective Action

However, from the matrix V, we can construct the following current

$$V^{-1}\partial_\alpha V = P_\alpha + Q_\alpha \tag{8}$$

which belongs to the Lie algebra of $O(d,d)$
From the symmetric space automorphism property of $O(d,d)/O(d)\times O(d)$, it follows that

$$P_\alpha^T = P_\alpha \ , \ \ Q_\alpha^T = -Q_\alpha$$

Therefore

$$P_\alpha = \frac{1}{2}[V^{-1}\partial_\alpha V + (V^{-1}\partial_\alpha V)^T] \tag{9}$$

$$Q_\alpha = \frac{1}{2}[V^{-1}\partial_\alpha V - (V^{-1}\partial_\alpha V)^T]$$

Two Dimensional String Effective Action

Now we can show that

$$Tr[\partial_\alpha M^{-1}\partial_\beta M] = -4\,Tr[P_\alpha P_\beta] \tag{10}$$

The currents in (9) and the action (1) are invariant under local gauge transformation

$$V \to Vh(x)$$

Consequently, the composite fields P_α and Q_α transform under $O(d) \times O(d)$ according to

$$P_\alpha \to h^{-1}(x) P_\alpha h(x)$$

$$Q_\alpha \to h^{-1}(x) Q_\alpha h(x) + h^{-1}(x)\partial_\alpha h(x)$$

This fact allows us to obtain the integrability conditions and the derivation of the general monodromy matrix

Sommaire

Classical Integrability and General Monodromy Matrix

We consider the case of the two dimensional σ model defined on the coset $O(d,d)/O(d)\times O(d)$. The integrability conditions following from the currents (8) correspond to the zero curvature condition, namely

$$\partial_\alpha[V^{-1}\partial_\beta V]-\partial_\beta[V^{-1}\partial_\alpha V]+[(V^{-1}\partial_\alpha V),(V^{-1}\partial_\beta V)]=0 \qquad (11)$$

Such equations with (9) are subject to the compatibility relations

$$\partial_\alpha Q_\beta-\partial_\beta Q_\alpha+[Q_\alpha,Q_\beta]=-[P_\alpha,P_\beta] \qquad (12)$$

$$D_\alpha P_\beta-D_\beta P_\alpha=0$$

with

$$D_\alpha P_\beta=\partial_\alpha P_\beta+[Q_\alpha,P_\beta]$$

Classical Integrability and General Monodromy Matrix

Let us consider the generalized current decomposition with arbitrary functions of the spectral parameter t as follows :

$$\widehat{V}^{-1}\partial_\alpha \widehat{V} = Q_\alpha + f(t)P_\alpha + g(t)\varepsilon_{\alpha\beta}P^\beta \tag{13}$$

where $f(t)$ and $g(t)$ are general functions satisfying the following conditions

$$f(t=0)=1, \lim_{t\to+\infty} f(t) = -1$$

$$g(t=0)=0, \lim_{t\to+\infty} g(t) = 0$$

Let us note that these functions have to possess singularities for $t=\pm 1$ Then, the integrability conditions (11) are rewritten as

$$\partial_\alpha[\widehat{V}^{-1}\partial_\beta\widehat{V}] - \partial_\beta[\widehat{V}^{-1}\partial_\alpha\widehat{V}] + [(\widehat{V}^{-1}\partial_\alpha\widehat{V}),(\widehat{V}^{-1}\partial_\beta\widehat{V})] = 0 \tag{14}$$

which can be expressed in terms of the functions f and g as follows :

$$\partial_\alpha Q_\beta - \partial_\beta Q_\alpha + 2[Q_\alpha, Q_\beta] + \partial_\alpha[f(t)+g(t)]P_\beta - \partial_\beta[f(t)+g(t)]P_\alpha$$

$$+[f(t)+g(t)] \times (\partial_\beta P_\alpha - \partial_\alpha P_\beta + 2[P_\alpha, P_\beta])$$

$$+4f(t)g(t)[P_\alpha, P_\beta] = 0$$

By using the relation (12), such equations take the following form :

$$(f' + g')[\partial_\alpha t P_\beta - \partial_\beta t P_\alpha] + (f+g)(\partial_\beta P_\alpha - \partial_\alpha P_\beta + 2[P_\alpha, P_\beta])$$

$$+(4fg - 1)[P_\alpha, P_\beta] + [Q_\alpha, Q_\beta] = 0$$

This leads to the integrability condition of the spectral parameter :

$$(\partial_\beta t)P_\alpha-(\partial_\alpha t)P_\beta=(f'+g')^{-1}\{(f+g)(\partial_\beta P_\alpha-\partial_\alpha P_\beta+2[P_\alpha,P_\beta]) +(4fg-1)[P_\alpha,P_\beta]+[Q_\alpha,Q_\beta]\}$$

which can be simplified by using the light-cone indices to the following equations :

$$\partial_\pm t=P_0^{-1}(P_\pm\pm H(t)) \tag{15}$$

Furthermore, in order to obtain the monodromy matrix in terms of generalized functions we can parametrize the $P_\pm$ quantities in terms of the vielbein as follows :

$$P_\pm=\begin{pmatrix}-E^{-1}\partial_\pm E & 0\\ 0 & E^{-1}\partial_\pm E\end{pmatrix}\quad,\quad Q_\pm=0 \tag{16}$$

Classical Integrability and General Monodromy Matrix

These currents that are subject to the integrability condition can be incorporated in the matrix $\widehat{V}$ as

$$\widehat{V}^{-1}\partial_{\pm}\widehat{V} = (f(t)+g(t))\begin{pmatrix} -E^{-1} & 0 \\ 0 & E^{-1} \end{pmatrix}\partial_{\pm}E \tag{17}$$

the matrices $E and G$ with are assumed to be diagonal, namely

$$E = diag\left(e^{\left(\frac{1}{2}\right)(\lambda+\psi_1)}, e^{\left(\frac{1}{2}\right)(\lambda+\psi_2)}, ..., e^{\left(\frac{1}{2}\right)(\lambda+\psi_d)} \right)$$

$$G = diag\left(e^{(\lambda+\psi_1)}, e^{(\lambda+\psi_2)}, ..., e^{(\lambda+\psi_d)} \right)$$

with $\Sigma\psi = 0$ so that $\lambda = \frac{1}{d} log\ detG$.

Classical Integrability and General Monodromy Matrix

The model contains a dilaton field ρ which is given in the conformal gauge by

$$\rho(x) = \rho_+(x^+) + \rho_-(x^-) = e^{-\overline{\Phi}}$$

with $\rho = det\ E$

where $\rho_+(x^+)$ and $\rho_-(x^-)$ are left and right moving solutions. The matter field equations of motion read

$$D^\alpha(\rho P_\alpha) = 0$$

where D_α is the $O(d) \times O(d)$ covariant derivative. There are two first order equations, namely

$$\partial_+\rho\partial_+\sigma = \frac{1}{2}\rho Tr(P_+P_+)$$

$$\partial_-\rho\partial_-\sigma = \frac{1}{2}\rho Tr(P_-P_-)$$

Classical Integrability and General Monodromy Matrix

where σ is defined by

$$\sigma \equiv log\lambda - \frac{1}{2}log(\partial_+\rho\partial_-\rho)$$

(8) can be recovered as the compatibility conditions of the linear system (Lax pair)

$$\widehat{V}^{-1}D_\alpha\widehat{V} = (f(t)+g(t))P_\alpha \ , \ \ Q_\alpha = 0 \tag{18}$$

where

$$D_\alpha\widehat{V} = \partial_\alpha\widehat{V} - \widehat{V}Q_\alpha$$

The consistency of (18) requires that the spectral parameter t itself be subject to a very similar system of linear differential equations as follows :

$$t^{-1}\partial_\pm t = (f(t)+g(t))\rho^{-1}\partial_\pm\rho$$

Indeed

$$\partial_\pm t = t(f(t) + g(t))e^{\overline{\Phi}}\partial_\pm e^{-\overline{\Phi}}$$

Thus, from the current form, we assume that

$$f(t) + g(t) = \sqrt{\frac{\omega - \rho_-}{\omega + \rho_+}}$$

where

$$\omega = -\rho_+ + \frac{e^{-\overline{\Phi}}}{1 - N^2(t)}$$

where $N(t)$ must verify that $N(t) \neq \pm 1$ with $N(t) = f(t) + g(t)$ Therefore, the generalized Monodromy Matrix has to be of the form

$$\widehat{M} = \widehat{V}(x,t)\widehat{V}^T\left(x, \frac{1}{t}\right) = \begin{pmatrix} M(\omega) & 0 \\ 0 & M^{-1}(\omega) \end{pmatrix}$$

Classical Integrability and General Monodromy Matrix

where $M(\omega)$ is diagonal with

$$M_i(\omega) = \frac{\omega_i - \omega}{\omega_i + \omega}$$

and

$$\omega_i = -\rho_+ + \frac{e^{-\overline{\Phi}}}{1 - N^2(t_i)}$$

- Finally this generalized monodromy matrix can be used to examine general currents and general integrability conditions in some cosmological models.
- We have shown that the generalized monodromy matrix transforms under the non compact T-duality group
- We have realized a connection between T-duality symmetry and general integrability properties by introducing a spectral parameter t which is space-time dependent

Thank you for your attention !