Double Field Theory and Membrane Sigma Models

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with Athanasios Chatzistavrakidis, Larisa Jonke and Richard J. Szabo

Fech Scen Khoo

Ruđer Bošković Institute

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Goal and Motivation

Goal:

Obtain a formulation of sigma model for double field theory (DFT)

Motivation:

• Quantum gravity $\rightarrow$ departure from Riemannian geometry: TM

• Generalized geometry (N. Hitchin): $TM \bigoplus T^*M$ ($O(d,d)$ symmetry)

Outline

- ▷ Introduction to double field theory (DFT)
- ▷ Our result: DFT sigma model (with an example)

Double Field Theory (DFT)

Introduction to double field theory

- A field theory, doubling the space of coordinates: contains both coordinates conjugate to momentum modes, and dual coordinates conjugate to winding modes of closed string
- $O(d,d;\mathbb{R})$-covariant formulation of the low-energy sector of string theory on a compact space
- T-duality ($O(d,d;\mathbb{Z})$ symmetry) is a symmetry of string theory that relates winding modes in a given compact space with momentum modes in another (dual) compact space.
- T-duality relates geometric and non-geometric fluxes

$$H_{ijk} \xleftrightarrow{\mathsf{T}_k} f_{ij}{}^k \xleftrightarrow{\mathsf{T}_j} Q_i{}^{jk} \xleftrightarrow{\mathsf{T}_i} R^{ijk} \ ,$$

where T_i denotes a T-duality transformation along $x^i \in M$.

Introduction to double field theory

Double field theory action (in doubled geometry),

$$S_{DFT} = \int d^{2d}X\, e^{-2\Phi}\mathcal{R}\ ,$$

where the Ricci scalar

$$\begin{aligned}\mathcal{R} &= \frac{1}{8}\mathcal{H}_{MN}\partial^M\mathcal{H}_{KL}(\partial^N\mathcal{H}^{KL} - 4\partial^L\mathcal{H}^{KN}) \\ &\quad -2\partial^M\Phi\,\partial^N\mathcal{H}_{MN} + 4\mathcal{H}_{MN}\partial^M\Phi\,\partial^N\Phi\end{aligned}$$

is expressed in generalized metric $\mathcal{H}_{MN}$, and dilaton Φ where $e^{-2\Phi} = \sqrt{|g|}e^{-2\phi}$ with spacetime metric g.
$M, N, K, L = 1, \ldots, 2d$

DFT action $S_{DFT} = S(\mathcal{H}_{MN}, \Phi)$, where generalized metric

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & -g^{ik}B_{kj} \\ B_{ik}g^{kj} & g_{ij} - B_{ik}g^{kl}B_{lj} \end{pmatrix}$$

with metric g_{ij} and Kalb Ramond B_{ij} field, $i, j, k, l = 1, \ldots, d$.

(M. Gualtieri, arXiv:math/0401221 [math.DG])

$\mathcal{H}^{MN} = \eta^{MP}\eta^{NQ}\mathcal{H}_{PQ}$, where the raising and lowering metric is

$$\eta_{MP} = \eta^{MP} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

which is an $O(d, d)$ metric.

Generalized diffeomorphisms:

Gauge transformation of the generalized metric results in the action of a generalized Lie derivative,

$$\begin{aligned}\delta_\xi \mathcal{H}_{MN} &= \mathsf{L}_\xi \mathcal{H}_{MN} \\ &= \xi^P \partial_P \mathcal{H}_{MN} + (\partial_M \xi^P - \partial^P \xi_M)\mathcal{H}_{PN} \\ &\quad +(\partial_N \xi^P - \partial^P \xi_N)\mathcal{H}_{MP} \ ,\end{aligned}$$

where $\partial_M = (\partial_\mu, \partial^\mu = 0), \mu = 1, \ldots, d$. $\partial^\mu = 0$ is a solution to a condition called

Strong Constraint: $\partial_M(\ldots)\partial^M(\ldots) = 0$

The dilaton transforms as a scalar tensor density, $\delta_\xi(e^{-2\Phi}) = \partial_M(\xi^M e^{-2\Phi})$.

When the strong constraint is imposed: $\partial^\mu = \frac{\partial}{\partial x_\mu} = 0$ (supergravity frame), S_{DFT} $(= \int d^{2d}X \, e^{-2\Phi}\mathcal{R})$ reduces to the Neveu-Schwarz sector of supergravity action in d dimensions,

$$S_{NS} = \int d^d X \sqrt{g} e^{-2\phi} (R + 4\partial_\mu \phi \, \partial^\mu \phi - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda}) \; .$$

(Hull and Zwiebach, arXiv:0904.4664 [hep-th])

A structure in double field theory

The C-bracket of vectors in double field theory is

$$\begin{aligned} [\![A,B]\!]^J &= \left(A^K\,\partial_K B^J - \tfrac{1}{2}\,A^K\,\partial^J B_K - \{A \leftrightarrow B\}\right) \\ &= \tfrac{1}{2}(\mathsf{L}_A B - \mathsf{L}_B A)^J \ , \end{aligned}$$

where the generalized Lie derivative in DFT is
$\mathsf{L}_A B = (A^I\,\partial_I B^J - B^I\,\partial_I A^J + B_I\,\partial^J A^I)e_J$.
Property: For generalized Lie derivatives of DFT,

$$\begin{aligned} &\big([\mathsf{L}_C, \mathsf{L}_A] - \mathsf{L}_{[\![C,A]\!]}\big)B \\ =\ &\eta_{IK}\,\eta_{JM}\,\big(B^J\,\partial^K C^M\,\partial^I A^L - B^J\,\partial^K A^M\,\partial^I C^L \\ &+ \tfrac{1}{2}\,C^M\,\partial^K A^J\,\partial^I B^L - \tfrac{1}{2}\,A^M\,\partial^K C^J\,\partial^I B^L\big)\,e_L\ . \end{aligned}$$

The right-hand side would vanish if we impose

$$\eta^{IJ}\,\partial_I f\,\partial_J g = 0\ , \qquad (0.1)$$

for all fields f, g of DFT. This condition (0.1) is known as the **strong constraint** in DFT.

Our proposal of a sigma model for DFT

Strategy:

From the well-known case of a Courant sigma model

(Liu, Weinstein and Xu, arXiv:dg-ga/9508013)

(Alexandrov, Kontsevich, Schwarz and Zaboronsky, arXiv:hep-th/9502010)

Double

$\downarrow$

Split

$\downarrow$

Project

A *large* Courant algebroid in generalized geometry

Step 1: Doubling the target space

Consider a target space T^*M, instead of M.

Definition.
Let $E \xrightarrow{\pi} T^*M$ be a vector bundle.
Let $[\cdot,\cdot]_E : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$.
Let $\langle \cdot,\cdot \rangle_E : \Gamma(E) \times \Gamma(E) \to C^\infty(M)$ be a symmetric $C^\infty(M)$-bilinear non-degenerate form.
Anchor map $\rho : E \to T(T^*M)$.
$\Rightarrow (E, [\cdot,\cdot]_E, \langle \cdot,\cdot \rangle_E, \rho)$ defines a *large* **Courant algebroid**. The structures satisfy certain properties (Leibniz rule, ...).

The vector bundle we consider is,

$$E = T(T^*M) \oplus T^*(T^*M) \ .$$

Introduction to a graded 2 manifold

A QP2-manifold is defined by $(\mathcal{M} = T^*[2]T[1]M, \omega, Q)$:

P-structure, ω: degree 2 symplectic structure
Q-structure, Q: degree 1 vector field (cohomological when $Q^2 = 0$)
these structures satisfy the compatibility condition: $\mathcal{L}_Q\omega = 0$

- Q gives rise to a degree 3 Hamiltonian function $\Theta \in C^\infty(\mathcal{M})$ as $Q = \{\Theta, \cdot\}$, with Poisson bracket of degree -2
- $Q^2 = 0$ implies the classical master equation

$$\{\Theta, \Theta\} = 0$$

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$$\{\Theta, \Theta\} = 0$$

$$\Downarrow$$

canonical Courant algebroid: $(E, \rho, [\cdot, \cdot]_E, \langle \cdot, \cdot \rangle_E)$

(Roytenberg, arXiv:math.SG/0203110)

Courant algebroid - QP2-manifold correspondence

On $\mathcal{M} = T^*[2]T[1]T^*M$ with a doubled target space, introduce local Darboux coordinates:

degree 0 function, $\mathbb{X}^I$: even
degree 1 $\mathbb{A}^{\hat{I}} = (\mathbb{A}^I, \widetilde{\mathbb{A}}_I)$: odd (anticommuting)
degree 2 $\mathbb{F}_I$: even

where $I = 1, \dots, 2d$ and $\hat{I} = 1, \dots, 4d$.

Graded Poisson bracket (degree -2):

$$\{\mathbb{F}_I, f(\mathbb{X})\} = \partial_I f \ , \quad \{\mathbb{A}^{\hat{I}}, \mathbb{A}^{\hat{J}}\} = \eta^{\hat{I}\hat{J}} \ , \quad \text{otherwise zero} \ ,$$

therefore the degree 2 symplectic structure,

$$\omega = \mathrm{d}\mathbb{F}_I \wedge \mathrm{d}\mathbb{X}^I + \frac{1}{2}\eta_{\hat{I}\hat{J}}\,\mathrm{d}\mathbb{A}^{\hat{I}} \wedge \mathrm{d}\mathbb{A}^{\hat{J}} \ .$$

Courant algebroid - QP2-manifold correspondence

The most general Hamiltonian function Θ given in these coordinates is a 3-degree,

$$\Theta = \rho^I{}_{\hat{I}}(\mathbb{X})\, \mathbb{F}_I\, \mathbb{A}^{\hat{I}} - \frac{1}{3!}\, T_{\hat{I}\hat{J}\hat{K}}(\mathbb{X})\, \mathbb{A}^{\hat{I}}\, \mathbb{A}^{\hat{J}}\, \mathbb{A}^{\hat{K}} \ , \qquad (0.2)$$

which gives

$$\begin{aligned} \{\Theta, \Theta\} \;=\; & (\eta^{\hat{I}\hat{J}}\, \rho^I{}_{\hat{I}}\, \rho^J{}_{\hat{J}})\, \mathbb{F}_I \mathbb{F}_J \\ & + (\rho^I{}_{\hat{I}}\, \partial_I \rho^J{}_{\hat{J}} - \rho^I{}_{\hat{J}}\, \partial_I \rho^J{}_{\hat{I}} - \eta^{\hat{K}\hat{L}}\, \rho^J{}_{\hat{K}}\, T_{\hat{L}\hat{I}\hat{J}}) \mathbb{A}^{\hat{I}} \mathbb{A}^{\hat{J}} \mathbb{F}_J \\ & - (\frac{1}{3}\, \rho^I{}_{\hat{L}}\, \partial_I\, T_{\hat{I}\hat{J}\hat{K}} + \frac{1}{4}\, \eta^{\hat{M}\hat{N}}\, T_{\hat{M}\hat{L}\hat{I}}\, T_{\hat{J}\hat{K}\hat{N}}) \mathbb{A}^{\hat{L}} \mathbb{A}^{\hat{I}} \mathbb{A}^{\hat{J}} \mathbb{A}^{\hat{K}} \ . \end{aligned}$$

$\{\Theta, \Theta\} = 0 \quad \Rightarrow \quad$ Courant algebroid properties in local expressions

From algebroid structure to a worldvolume description I

The *large* Courant sigma model is

$$S[\mathbb{X},\mathbb{A},\mathbb{F}] \;=\; \int_{\Sigma_3} \Big(\mathbb{F}_I\wedge \mathrm{d}\mathbb{X}^I+\tfrac12\,\eta_{\hat I\hat J}\,\mathbb{A}^{\hat I}\wedge \mathrm{d}\mathbb{A}^{\hat J}-\rho^I{}_{\hat I}(\mathbb{X})\,\mathbb{A}^{\hat I}\wedge\mathbb{F}_I$$
$$+\tfrac16\,T_{\hat I\hat J\hat K}(\mathbb{X})\,\mathbb{A}^{\hat I}\wedge\mathbb{A}^{\hat J}\wedge\mathbb{A}^{\hat K}\Big)\ ,$$

where the map

$$\mathbb{X}:\Sigma_3\longrightarrow T^*M\ ,$$

with local coordinates (x^i,p_i) in the target space T^*M.
The components of this map are

$$\mathbb{X}=(\mathbb{X}^I)=(\mathbb{X}^i,\mathbb{X}_i)=:(X^i,\widetilde{X}_i)\ ,$$

where the fields X^i and $\widetilde{X}_i$ are identified with the pullbacks of the coordinate functions, $X^i=\mathbb{X}^*(x^i)$ and $\widetilde{X}_i=\mathbb{X}^*(p_i)$, with $i=1,\dots,d$, $I=1,\dots,2d$ and $\hat I=1,\dots,4d$.

From algebroid structure to a worldvolume description II

The *large* Courant sigma model

$$S[\mathbb{X},\mathbb{A},\mathbb{F}] = \int_{\Sigma_3} \Big(\mathbb{F}_I \wedge \mathrm{d}\mathbb{X}^I + \tfrac{1}{2}\,\eta_{\hat{I}\hat{J}}\,\mathbb{A}^{\hat{I}} \wedge \mathrm{d}\mathbb{A}^{\hat{J}} - \rho^I{}_{\hat{I}}(\mathbb{X})\,\mathbb{A}^{\hat{I}} \wedge \mathbb{F}_I + \tfrac{1}{6}\,T_{\hat{I}\hat{J}\hat{K}}(\mathbb{X})\,\mathbb{A}^{\hat{I}} \wedge \mathbb{A}^{\hat{J}} \wedge \mathbb{A}^{\hat{K}} \Big) ,$$

for $I = 1,\dots,2d$ and algebroid index $\hat{I} = 1,\dots,4d$.

The sections of the bundle, $\Gamma(E)$: $(\mathbb{A}^{\hat{I}}) = (\mathbb{A}^I, \widetilde{\mathbb{A}}_I) = (\mathbb{A}^i, \mathbb{A}_i, \widetilde{\mathbb{A}}_i, \widetilde{\mathbb{A}}^i)$,
where $\mathbb{A} = \mathbb{A}_V + \mathbb{A}_F := \mathbb{A}^I\,\partial_I + \widetilde{\mathbb{A}}_I\,\mathrm{d}\mathbb{X}^I$.
The basis vectors on T^*M: $(\partial_I) = (\partial/\partial X^i, \partial/\partial\widetilde{X}_i) =: (\partial_i, \tilde{\partial}^i)$
The basis forms on T^*M: $(\mathrm{d}\mathbb{X}^I) := (\mathrm{d}X^i, \mathrm{d}\widetilde{X}_i)$
For the anchor $\rho^I{}_{\hat{J}}$, the components are $(\rho^I{}_J, \widetilde{\rho}^{IJ})$.

$\therefore$ Defined in terms of the structures of *large* Courant algebroid: ρ, $[\cdot,\cdot]_E, \langle\cdot,\cdot\rangle_E$

From algebroid structure to a worldvolume description III

The *large* Courant sigma model

$$S[\mathbb{X},\mathbb{A},\mathbb{F}] \;=\; \int_{\Sigma_3} \Big(\mathbb{F}_I \wedge \mathrm{d}\mathbb{X}^I + \tfrac{1}{2}\,\eta_{\hat{I}\hat{J}}\,\mathbb{A}^{\hat{I}} \wedge \mathrm{d}\mathbb{A}^{\hat{J}} - \rho^I{}_{\hat{I}}(\mathbb{X})\,\mathbb{A}^{\hat{I}} \wedge \mathbb{F}_I \\ + \tfrac{1}{6}\,T_{\hat{I}\hat{J}\hat{K}}(\mathbb{X})\,\mathbb{A}^{\hat{I}} \wedge \mathbb{A}^{\hat{J}} \wedge \mathbb{A}^{\hat{K}}\Big) \ ,$$

for $I = 1,\ldots,2d$ and $\hat{I} = 1,\ldots,4d$.

Worldvolume 1-form $\mathbb{A} \in \Omega^1\big(\Sigma_3,\mathbb{X}^*\big(T(T^*M) \oplus T^*(T^*M)\big)\big)$.

Auxiliary worldvolume 2-form $\mathbb{F} \in \Omega^2(\Sigma_3,\mathbb{X}^*T^*(T^*M))$.

The twist is decomposed as

$$T_{\hat{I}\hat{J}\hat{K}} := \begin{pmatrix} A_{IJK} & B_{IJ}{}^K \\ C_I{}^{JK} & D^{IJK} \end{pmatrix} \ .$$

Note: The *large* Courant sigma model has an $O(2d,2d)$ metric, while double field theory has an $O(d,d)$ metric.

Introduce a decomposition

Step 2: Splitting the bundle

Recall that $(\mathbb{A}^{\hat{I}}) = (\mathbb{A}^I, \widetilde{\mathbb{A}}_I)$, and $\rho^I{}_{\hat{J}} = (\rho^I{}_J, \widetilde{\rho}^{IJ})$.
Decompose the sections of the bundle, basis, and anchor in

$$\mathbb{A}^I_\pm = \tfrac{1}{2}\left(\mathbb{A}^I \pm \eta^{IJ}\,\widetilde{\mathbb{A}}_J\right) , \quad e^\pm_I = \partial_I \pm \eta_{IJ}\,\mathrm{d}\mathbb{X}^J ,$$

$$(\rho_\pm)^I{}_J = \rho^I{}_J \pm \eta_{JK}\,\widetilde{\rho}^{IK} ,$$

where an $O(d,d)$ metric η is employed.
$\therefore$ The vector bundle is found to decompose as

$$E = T(T^*M) \oplus T^*(T^*M) = L_+ \oplus L_- ,$$

where $L_\pm$ is the bundle whose space of sections, $\mathbb{A}^I_\pm$ is spanned locally by $e^\pm_I$.
Anchors $(\rho_\pm)^I{}_J$: $L_\pm \to T(T^*M)$ on the doubled space.

Introduce a decomposition

When the *large* Courant sigma model

$$S[\mathbb{X},\mathbb{A},\mathbb{F}] \;=\; \int_{\Sigma_3} \Big(\mathbb{F}_I \wedge \mathrm{d}\mathbb{X}^I + \tfrac{1}{2}\,\eta_{\hat{I}\hat{J}}\,\mathbb{A}^{\hat{I}} \wedge \mathrm{d}\mathbb{A}^{\hat{J}} - {\rho^I}_{\hat{I}}(\mathbb{X})\,\mathbb{A}^{\hat{I}} \wedge \mathbb{F}_I \\ + \tfrac{1}{6}\,T_{\hat{I}\hat{J}\hat{K}}(\mathbb{X})\,\mathbb{A}^{\hat{I}} \wedge \mathbb{A}^{\hat{J}} \wedge \mathbb{A}^{\hat{K}}\Big)$$

is expressed in terms of $\mathbb{A}^I_\pm$ and ${(\rho_\pm)^I}_J$,

$$= \int_{\Sigma_3} \Big(\mathbb{F}_I \wedge \mathrm{d}\mathbb{X}^I + \eta_{IJ}\,(\mathbb{A}^I_+ \wedge \mathrm{d}\mathbb{A}^J_+ - \mathbb{A}^I_- \wedge \mathrm{d}\mathbb{A}^J_-) \\ -\big({(\rho_+)^I}_K\,\mathbb{A}^K_+ + {(\rho_-)^I}_K\,\mathbb{A}^K_-\big) \wedge \mathbb{F}_I \\ + \tfrac{1}{6}\,T_{IJK}\,\mathbb{A}^I_+ \wedge \mathbb{A}^J_+ \wedge \mathbb{A}^K_+ + \tfrac{1}{2}\,T'_{IJK}\,\mathbb{A}^I_- \wedge \mathbb{A}^J_+ \wedge \mathbb{A}^K_+ \\ + \tfrac{1}{2}\,T''_{IJK}\,\mathbb{A}^I_+ \wedge \mathbb{A}^J_- \wedge \mathbb{A}^K_- + \tfrac{1}{6}\,T'''_{IJK}\,\mathbb{A}^I_- \wedge \mathbb{A}^J_- \wedge \mathbb{A}^K_-\Big)\,,$$

where the components of T, T', T'', T''' are combinations of the twist components $A_{IJK}, {B_{IJ}}^K, {C_I}^{JK}, D^{IJK}$ in $T_{\hat{I}\hat{J}\hat{K}}$.

Step 3: Projecting to a subbundle

- Project with the map $\mathrm{p}_+ : E \longrightarrow L_+$, i.e. $\mathbb{A}^I_- = 0, (\rho_-)^I{}_J = 0$.
- Identify $\mathbb{A}^I_+ = A^I$ and $\mathbb{F}_I = F_I$.

We obtain the $O(d,d)$ invariant **DFT membrane sigma model** (topological sector),

$$S[\mathbb{X}, A, F] \;=\; \int_{\Sigma_3} \Big(F_I \wedge \mathrm{d}\mathbb{X}^I + \eta_{IJ}\, A^I \wedge \mathrm{d}A^J - (\rho_+)^I{}_J\, A^J \wedge F_I + \tfrac{1}{6}\, T_{IJK}\, A^I \wedge A^J \wedge A^K\Big)\ ,$$

where $I = 1, \dots, 2d$. The $O(d,d)$ metric is

$$\eta_{IJ} = \begin{pmatrix} 0 & \delta_i{}^j \\ \delta^i{}_j & 0 \end{pmatrix}\ .$$

Double field theory sigma model

Summary:

$$\mathsf{p}_+ : E \longrightarrow L_+ \ , \quad (\mathbb{A}_V, \mathbb{A}_F) \longmapsto \mathbb{A}_+ := A \ ,$$

$$\therefore \ \mathsf{p}_+(\mathbb{A}) = \mathbb{A}_+ = \mathbb{A}_+^I \, e_I^+ \ .$$

A is identified as the pullback of a DFT vector.
DFT sigma model,

$$S[\mathbb{X}, A, F] \;=\; \int_{\Sigma_3} \big(F_I \wedge \mathrm{d}\mathbb{X}^I + \eta_{IJ}\, A^I \wedge \mathrm{d}A^J - (\rho_+)^I{}_J\, A^J \wedge F_I + \tfrac{1}{6}\, T_{IJK}\, A^I \wedge A^J \wedge A^K \big) \ .$$

Fluxes in Double Field Theory

DFT fluxes

Given the components of the anchor in the Kalb-Ramond 2-form field B and the bivector field β,

$$(\rho_+)^I{}_J = \begin{pmatrix} \delta^i{}_j & \beta^{ij} \\ B_{ij} & \delta_i{}^j + \beta^{jk}\, B_{ki} \end{pmatrix} , \tag{0.3}$$

where the anchor **satisfies** $(\rho_+)^K{}_I\, \eta^{IJ}\, (\rho_+)^L{}_J = \eta^{KL}$, we can derive DFT fluxes from the topological part of the DFT sigma model (with an untwisted C-bracket),

$$S = \int_{\Sigma_3} \left(F_I \wedge (\mathrm{d}\mathbb{X}^I - (\rho_+)^I{}_J\, A^J) + \eta_{IJ}\, A^I \wedge \mathrm{d}A^J \right) .$$

Taking the equation of motion for the auxiliary 2-form F_I, we obtain $\mathrm{d}\mathbb{X}^I = (\rho_+)^I{}_J\, A^J$ which implies

$$A^I = (\rho_+)_J{}^I\, \mathrm{d}\mathbb{X}^J .$$

DFT fluxes

Eliminating F_I, the action becomes

$$\int_{\partial\Sigma_3} \left(\eta_{IJ}\,(\rho_+)_K{}^I\,A^J \wedge \mathrm{d}\mathbb{X}^K\right)$$
$$+ \int_{\Sigma_3} \eta_{IM}\,(\rho_+)^L{}_K\,(\rho_+)_N{}^M\,\partial_L(\rho_+)^N{}_J\,A^I \wedge A^J \wedge A^K \ .$$

The three-dimensional term in this action encodes the DFT fluxes $\hat{T}$ which satisfy

$$2\,\rho^K{}_{[L}\,\partial_K\rho^I{}_{M]} - \rho_{K[L}\,\partial^I\rho^K{}_{M]} = \rho^I{}_J\,\eta^{JK}\,\hat{T}_{KLM}\ , \qquad (0.4)$$

where $\rho = \rho_+$ in notation.

DFT fluxes

Recall/Check that the 4 types of fluxes (H, f, Q, R) which are related by T-duality, in a holonomic frame read as

$$
\begin{aligned}
H_{ijk} &= 3\,\partial_{[i} B_{jk]} + 3\, B_{[i\underline{l}}\, \tilde{\partial}^l B_{jk]} \ , \\
f_{ij}{}^k &= \tilde{\partial}^k B_{ij} + \beta^{kl}\, H_{lij} \ , \\
Q_k{}^{ij} &= \partial_k \beta^{ij} + B_{kl}\, \tilde{\partial}^l \beta^{ij} + 2\, \beta^{l[i}\, \tilde{\partial}^{j]} B_{lk} + \beta^{il}\, \beta^{jm}\, H_{lmk} \ , \\
R^{ijk} &= 3\, \tilde{\partial}^{[i} \beta^{jk]} + 3\, \beta^{[i\underline{l}}\, \partial_l \beta^{jk]} \\
&\quad + 3\, B_{lm}\, \beta^{[i\underline{l}}\, \tilde{\partial}^m \beta^{jk]} + 3\, \beta^{[i\underline{l}}\, \beta^{j\underline{m}}\, \tilde{\partial}^{k]} B_{lm} + \beta^{il}\, \beta^{jm}\, \beta^{kn}\, H_{lmn} \ .
\end{aligned}
$$

Bianchi identities for DFT fluxes (from gauge invariance)

DFT topological sigma model,

$$S[\mathbb{X},A,F] \;=\; \int_{\Sigma_3} \big(F_I\wedge(\mathrm{d}\mathbb{X}^I-\rho^I{}_J\,A^J)+\eta_{IJ}\,A^I\wedge\mathrm{d}A^J + \tfrac13\,\hat{T}_{IJK}\,A^I\wedge A^J\wedge A^K\big)\ .$$

Consider the infinitesimal gauge transformations,

$$\delta_\epsilon\mathbb{X}^I=\rho^I{}_J(\mathbb{X})\,\epsilon^J\ , \qquad \delta_\epsilon A^I=\mathrm{d}\epsilon^I+\eta^{IJ}\,\hat{T}_{JKL}(\mathbb{X})\,A^K\,\epsilon^L\ ,$$

where ϵ is a gauge parameter (a function only of the worldvolume coordinates on Σ_3). Let the worldvolume derivative

$$D\mathbb{X}^I=\mathrm{d}\mathbb{X}^I-\rho^I{}_J(\mathbb{X})\,A^J\ ,$$

it transforms as

$$\begin{aligned}\delta_\epsilon D\mathbb{X}^I &= \epsilon^J\,\partial_K\rho^I{}_J\,D\mathbb{X}^K+\big(2\rho^K{}_{[L}\,\partial_K\rho^I{}_{M]}-\rho^I{}_J\,\eta^{JK}\,\hat{T}_{KLM}\big)\,A^L\,\epsilon^M\\ &= \epsilon^J\,\partial_K\rho^I{}_J\,D\mathbb{X}^K+\rho_{K[L}\,\partial^I\rho^K{}_{M]}\,A^L\,\epsilon^M\ .\end{aligned}$$

The derivative transforms covariantly if the last term vanishes.

Bianchi identities for DFT fluxes (from gauge invariance)

The gauge variation of the sigma model gives

$$
\begin{aligned}
\delta_\epsilon S = \int_{\Sigma_3} \Big(& \eta_{IJ}\, \mathrm{d}\epsilon^I \wedge \mathrm{d}A^J + \rho_{K[L}\, \partial^I \rho^K{}_{M]}\, \epsilon^M\, F_I \wedge A^L \\
& + \delta_\epsilon F_K \wedge D\mathbb{X}^K + \epsilon^J \left(\partial_K \rho^I{}_J\, F_I - \partial_K \hat{T}_{ILJ}\, A^I \wedge A^L\right) \wedge D\mathbb{X}^K \\
& + \epsilon^L \big(\eta^{MN}\, \hat{T}_{MJK}\, \hat{T}_{ILN} + \rho^M{}_I\, \partial_M \hat{T}_{KJL} + \tfrac{1}{3}\, \rho^M{}_L\, \partial_M \hat{T}_{IJK}\big) \\
& \qquad A^I \wedge A^J \wedge A^K \Big) \ .
\end{aligned}
$$

The first term is a total derivative.
To cancel the second term, one may **impose** the following constraint:

$$
\begin{aligned}
& \rho_{KL}\, \partial^I \rho^K{}_M\, \epsilon^M\, F_I \wedge A^L \\
= \ & \rho_{KL}\, \partial_i \rho^K{}_M\, \epsilon^M\, F^i \wedge A^L + \rho_{KL}\, \tilde{\partial}^i \rho^K{}_M\, \epsilon^M\, F_i \wedge A^L \\
= \ & 0 \ .
\end{aligned} \tag{0.5}
$$

This requirement is related to the strong constraint.

Bianchi identities for DFT fluxes (from gauge invariance)

The second line in $\delta_\epsilon S$ vanishes by postulating the gauge variation of the auxiliary 2-form F_I as

$$\delta_\epsilon F_K = -\epsilon^J \left(\partial_K \rho^I{}_J\, F_I - \partial_K \hat{T}_{ILJ}\, A^I \wedge A^L\right) .$$

An additional requirement is the vanishing of the last term in $\delta_\epsilon S$:

$$3\,\eta^{MN}\,\hat{T}_{M[JK}\,\hat{T}_{I]LN} + 3\,\rho^M{}_{[I}\,\partial_{\underline{M}}\hat{T}_{KJ]L} + \rho^M{}_L\,\partial_M \hat{T}_{IJK} = 0 ,$$

which can be rewritten into

$$3\,\eta^{MN}\,\hat{T}_{M[JK}\,\hat{T}_{IL]N} + 4\,\rho^M{}_{[I}\,\partial_{\underline{M}}\hat{T}_{KJL]} = 0 . \qquad (0.6)$$

This requirement gives the **Bianchi identities** in DFT.

Bianchi identities for DFT fluxes (from gauge invariance)

Substitution into (0.6) of the DFT fluxes (H, f, Q, R) together with the anchor

$$(\rho_+)^I{}_J = \begin{pmatrix} \delta^i{}_j & \beta^{ij} \\ B_{ij} & \delta_i{}^j + \beta^{jk}\, B_{ki} \end{pmatrix}$$

leads to

$$\begin{aligned}
\mathcal{D}_{[i} H_{jkl]} &= \tfrac{3}{2}\, H_{m[ij}\, f_{kl]}{}^m \ , \\
\mathcal{D}_{[i} f_{jk]}{}^l - \tfrac{1}{3}\, \widetilde{\mathcal{D}}^l H_{ijk} &= Q_{[i}{}^{lm}\, H_{jk]m} - f_{[ij}{}^m\, f_{k]m}{}^l \ , \\
\mathcal{D}_{[i} Q_{j]}{}^{kl} + \widetilde{\mathcal{D}}^{[k} f_{ij}{}^{l]} &= \tfrac{1}{2}\, f_{ij}{}^m\, Q_m{}^{kl} + \tfrac{1}{2}\, H_{ijm}\, R^{mkl} - 2\, Q_{[i}{}^{m[k}\, f_{j]m}{}^{l]} \ , \\
\widetilde{\mathcal{D}}^{[i} Q_l{}^{jk]} - \tfrac{1}{3}\, \mathcal{D}_l R^{ijk} &= f_{lm}{}^{[i}\, R^{jk]m} - Q_m{}^{[ij}\, Q_l{}^{k]m} \ , \\
\widetilde{\mathcal{D}}^{[i} R^{jkl]} &= \tfrac{3}{2}\, R^{m[ij}\, Q_m{}^{kl]} \ ,
\end{aligned}$$

where

$$\mathcal{D}_i = \partial_i + B_{ji}\, \tilde{\partial}^j \qquad \text{and} \qquad \tilde{\mathcal{D}}^i = \tilde{\partial}^i + \beta^{ji}\, \mathcal{D}_j \ .$$

The strong constraint for gauge invariance I

Recall the imposition of the following constraint for the DFT sigma model to be gauge invariant,

$$\begin{aligned} & \rho_{KL}\,\partial^I\rho^K{}_M\,\epsilon^M\,F_I\wedge A^L \\ = & \ \rho_{KL}\,\partial_i\rho^K{}_M\,\epsilon^M\,F^i\wedge A^L+\rho_{KL}\,\tilde{\partial}^i\rho^K{}_M\,\epsilon^M\,F_i\wedge A^L \\ = & \ 0\ . \end{aligned}$$

This can be solved by having $\tilde{\partial}^i = 0$ and $F^i = 0$, i.e. eliminating dual coordinates. Or alternatively, $\partial_i = 0$ and $F_i = 0$, or other mixed choice.

The solutions can be noticed from the combinations below:
(i) The strong constraint required for the closure of the C-bracket, $[\mathrm{L}_C, \mathrm{L}_A] = \mathrm{L}_{[\![C,A]\!]}$:

$$\eta^{IJ}\partial_I f\partial_J g = 0 = \delta_i^{\,j}\tilde{\partial}^i f\partial_j g + \delta^i{}_j\partial_i f\tilde{\partial}^j g\ .$$

This is solved by having either $\tilde{\partial}^i = 0$ or $\partial_i = 0$.

The strong constraint for gauge invariance I

(ii) Recall one of the conditions from the graded manifold correspondence to Courant algebroid: $\eta^{\hat{I}\hat{J}}\,\rho^I{}_{\hat{I}}\,\rho^J{}_{\hat{J}}\mathbb{F}_I\mathbb{F}_J = 0$.
After projections, we get eventually

$$\left(\rho^K{}_I\,\eta^{IJ}\,\rho^L{}_J\right) F_K\,F_L = \eta^{KL}\,F_K\,F_L =: F^K\,F_K\ .$$

For this to vanish, it is solved by either $F^i = 0$ or $F_i = 0$.
Note that in DFT,

$$\rho^K{}_I\,\eta^{IJ}\,\rho^L{}_J = \eta^{KL}\ ,$$

it is non-vanishing on the right hand side. This can be checked explicitly from the choice of anchor $(\rho_+)^I{}_J = \begin{pmatrix} \delta^i{}_j & \beta^{ij} \\ B_{ij} & \delta_i{}^j + \beta^{jk}\,B_{ki} \end{pmatrix}$ that parametrizes the DFT fluxes.

The strong constraint for gauge invariance II

Previously, the requirement that $3\,\eta^{MN}\,\hat{T}_{M[JK}\,\hat{T}_{IL]N} + 4\,\rho^{M}{}_{[I}\,\partial_{\underline{M}}\hat{T}_{KJL]} = 0$ for gauge invariance, gives Bianchi identities for the fluxes. There is an indication of strong constraint.

We could impose

$$3\,\eta^{MN}\,\hat{T}_{M[JK}\,\hat{T}_{IL]N} + 4\,\rho^{M}{}_{[I}\,\partial_{\underline{M}}\hat{T}_{KJL]} = \mathcal{Z}_{IJKL}\ ,$$

where $\mathcal{Z}$ is a 4-form. We substitute the DFT flux expression, $2\,\rho^{K}{}_{[L}\,\partial_{K}\rho^{I}{}_{M]} - \rho_{K[L}\,\partial^{I}\rho^{K}{}_{M]} = \rho^{I}{}_{J}\,\eta^{JK}\,\hat{T}_{KLM}$. Upon contracting with $A\wedge A\wedge A\wedge A$, and using $\rho^{K}{}_{I}\,\eta^{IJ}\,\rho^{L}{}_{J} = \eta^{KL}$, we get

$$\mathcal{Z}_{IJKL}A^{I}A^{J}A^{L}A^{K} = 3\,\rho_{NJ}\,\rho_{QI}\,(\partial_{M}\rho^{N}{}_{K})\,(\partial^{M}\rho^{Q}{}_{L})\,A^{I}A^{J}A^{L}A^{K}$$

in contracted derivatives. This term vanishes if we impose the strong constraint: $\eta^{IJ}\,\partial_{I}f\,\partial_{J}g = 0$.

Dynamics in the double field theory sigma model

Closed string dynamics in the boundary of the open membrane

We add a symmetric boundary term to the topological doubled sigma model,

$$S[\mathbb{X},A,F] = \int_{\Sigma_3} \left(F_I \wedge \mathrm{d}\mathbb{X}^I + \eta_{IJ}\, A^I \wedge \mathrm{d}A^J - (\rho_+)^I{}_J\, A^J \wedge F_I\right) + \int_{\Sigma_3} \tfrac{1}{6}\, T_{IJK}\, A^I \wedge A^J \wedge A^K + \int_{\partial\Sigma_3} \tfrac{1}{2}\, g_{IJ}(\mathbb{X})\, A^I \wedge *A^J \;, \qquad (0.7)$$

where in general $(\rho_+)^I{}_J = \begin{pmatrix} \rho^i{}_j & \rho^{ij} \\ \rho_{ij} & \rho_i{}^j \end{pmatrix}$, $A^I = (q^i, p_i)$,

$$T_{IJK} = \begin{pmatrix} H_{ijk} & f_{ij}{}^k \\ Q_i{}^{jk} & R^{ijk} \end{pmatrix} \;,\; g_{IJ} = \begin{pmatrix} g_{ij} & g_i{}^j \\ g^i{}_j & g^{ij} \end{pmatrix} \;.$$

The worldsheet theories for the 4 T-dual closed string backgrounds with constant H-, f-, Q- and R-fluxes can be derived from the DFT sigma model.

Example: NS–NS (Neveu-Schwarz) flux

To describe the geometric H-flux frame on the 3-torus, we choose the data

$$(\rho_+)^I{}_J = \begin{pmatrix} \delta^i{}_j & 0 \\ 0 & 0 \end{pmatrix} \ , T_{IJK} = \begin{pmatrix} H_{ijk} & 0 \\ 0 & 0 \end{pmatrix} \ , g_{IJ} = \begin{pmatrix} 0 & 0 \\ 0 & g^{ij} \end{pmatrix} \ ,$$

where g^{ij} denotes a constant metric with inverse g_{ij}. Then the membrane action becomes

$$\begin{aligned} S_{\text{DFT}} &= \int_{\Sigma_3} \big(F_I \wedge \mathrm{d}\mathbb{X}^I + q^i \wedge \mathrm{d}p_i + p_i \wedge \mathrm{d}q^i - q^i \wedge F_i \\ &\quad + \tfrac{1}{6}\, H_{ijk}\, q^i \wedge q^j \wedge q^k \big) + \int_{\partial\Sigma_3} \tfrac{1}{2}\, g^{ij}\, p_i \wedge * p_j \ . \end{aligned} \tag{0.8}$$

We are interested in the on-shell membrane theory. The equation of motion for F_I yields two relations, one from F_i and the other from F^i, giving

$$q^i = \mathrm{d}X^i \qquad \text{and} \qquad \mathrm{d}\widetilde{X}_i = 0 \ (\therefore \text{dual coordinates removed}) \ .$$

Example: NS–NS flux

The on-shell action takes the form

$$\int_{\partial\Sigma_3} \left(p_i \wedge \mathrm{d}X^i + \tfrac{1}{2}\, g^{ij}\, p_i \wedge *p_j\right) + \int_{\Sigma_3} \tfrac{1}{6}\, H_{ijk}\, \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k \ .$$

After integrating out p_i using $*^2 = 1$, it becomes

$$S_H[X] := \int_{\partial\Sigma_3} \tfrac{1}{2}\, g_{ij}\, \mathrm{d}X^i \wedge *\mathrm{d}X^j + \int_{\Sigma_3} \tfrac{1}{6}\, H_{ijk}\, \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k$$

for the closed string sigma model on $\partial\Sigma_3$ with 3-torus target space and NS–NS flux.

On the strong constraint in DFT

• Its imposition preserves the gauge invariance of the DFT membrane sigma model, giving also Bianchi identities for the fluxes.

• Learned from the C-bracket in DFT: it facilitates the closure of generalized Lie derivatives, hence the closure of gauge transformations.

• It controls the violation of two of the axioms in Courant algebroid, i.e. homormorphism and Jacobi identity.

Conclusions

▷ Method of Doubling-Splitting-Projecting:

Large Courant algebroid $\xrightarrow{p_+}$ Double field theory $\xrightarrow{\text{strong constraint}}$ canonical Courant algebroid

- geometric origin of double field theory

▷ Fluxes and their Bianchi identities in double field theory can be derived from the sigma model.

▷ A double field theory sigma model that upon choosing an anchor ρ and flux T, and adding a suitable symmetric term in the boundary of the membrane, captures geometric and non-geometric flux background descriptions, and motion of a closed string in the boundary.

Thank You