Quantum space and quantum completeness

Conference on Symmetries, Geometry and Quantum Gravity, Primošten 2018

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June 18, 2018

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Singularity?

- Intuitively, a spacetime singularity is a "place" where the curvature "blows up" or other "pathological behavior" of the metric takes place [Wald].
- The difficulty in making this notion into a satisfactory one is why some words in the previous sentence are left in quotes.
- Classification of classical singularities (regular/singular points, quasi-regular, (non)scalar singularities) is presented in

G. F. R. Ellis and B. G. Schmidt, "Singular space-times," Gen. Rel. Grav. **8** (1977) 915.

Geodesic completeness

- In general relativity (GR) it is often to encounter solutions with singularity, i.e. black holes. The presence of singularity is equivalent to the statement that the corresponding metric is geodesically incomplete.

- Geodesics are generalization of straight lines in curved space, and therefore describe the motion of the free falling test particles.

- A spacetime is called geodesically complete if every maximal geodesic is defined on an entire real line. If a spacetime is (time-like) geodesically incomplete, then the evolution of certain test particles is not well defined after a finite proper time and is said to be singular.

S.W. Hawking, R. Penrose, "The singularities of gravitational collapse and cosmology," Proceedings of the Royal Society London A, 314, 529 (1970)

Quantum completeness

- How can one formulate a condition in quantum theory which would determine whether or not a certain spacetime is singular?
- One way is to look at the expectation value of certain "physical operators" and see whether they diverge or not.
- Another way is to analyze under which condition is the evolution of any state uniquely defined for all time. Such system is non-singular or quantum-mechanically complete. If this is not the case, then one looses predictability and the system is said to be singular.

Quantum completeness

- Wald(1979) and Horowitz&Marolf(1995) connected the quantum completeness with the unique SAE of the Hamiltonian(wave operator) of the corresponding equation of motion for the test field.
- In order to determine whether the symmetric operator $\mathcal{L}$ is essentially self-adjoint, or if it admits SAE it is enough to investigate the square integrable solutions of the eigenvalue equations with purely imaginary eigenvalues.

$$\mathcal{L}[f] = \lambda f \;\Rightarrow\; \lambda \longrightarrow \pm i \Rightarrow \;\; \mathcal{L}_{\pm}[f] = \pm i f$$

This way we are searching for a possible domain in which $\mathcal{L}$ could have imaginary eigenvalues, that is we are searching for domains on which $\mathcal{L}$ is definitely not self-adjoint, because a self-adjoint operator has only real eigenvalues.

SAE

In the theory of SAE developed by von Neumann one is interested in the number of linearly independent square integrable solutions with eigenvalues $+i$ and $-i$. These numbers, called deficiency indices (n_+, n_-) determine the following:

1. if $(n_+, n_-) = (0, 0)$, then $\mathcal{L}$ is essentially self-adjoint
2. if $n_+ = n_- \equiv n$, then $\mathcal{L}$ is not self-adjoint, but admits SAE, and the new domain on which it is self-adjoint is given by

   $$D^{SAE}(\mathcal{L}) = D(\mathcal{L}) \oplus \{f_+ + U_n f_-\}$$

   where U_n is $n \times n$ unitary matrix.
3. if $n_+ \neq n_-$, then $\mathcal{L}$ does not admits SAE

Simple example

Spherical symmetric metric

$$ds^2 = dr^2 + f^2(r) d\Omega_n$$

where $d\Omega_n$ is the standard metric on the n-sphere. We investigate

$$\Delta_n \psi = \pm i \psi$$

Now, using the separation of variables $\psi = R(r)Y(\Omega)$ one obtains the radial equation

$$R'' + \frac{nf'}{f} R' - \frac{c}{f^2} R = \pm i R$$

where $c \geq 0$ is an eigenvalue of the Laplacian on the n-sphere.

The self-adjointness is equivalent to the statement that for each c and each choice of the eigenvalue $\pm i$, there are no square-integrable solutions near the origin.

Simple example

One can look only at the case $c = 0$, since $c > 0$ only increases the divergence of the solution at the origin $r = 0$.

If $f(r) = r^k$ near the origin, the solution is given by $R(r) = r^{1-nk}$, and there are no square-integrable solutions (with respect to the measure $d^{n+1}x\sqrt{-g} \longrightarrow r^{kn}dr$) if

$$k \geq \frac{3}{n}.$$

All such metrics with $f(r) = r^k$ $(k \geq \frac{3}{n})$ near the origin are quantum-mechanically nonsingular.

Only for $k = 1$ we have geodesically complete metric, so we see that even in this simple example we found a large class of classically singular geometries, which are quantum-mechanically complete.

BTZ (Banados, Teitelboim, Zanelli)

BTZ black hole is a solution of GR in $2+1$. For $J=0$ we have

$$g_{\mu\nu} = \begin{pmatrix} \frac{r^2}{l^2} - 8GM & 0 & 0 \\ 0 & -\frac{1}{\frac{r^2}{l^2} - 8GM} & 0 \\ 0 & 0 & -r^2 \end{pmatrix}$$

Since gravity in $2+1$ has no propagating degrees of freedom, it is simple, has a lot of analytical results (unlike 3+1 case) and therefor presents a good “toy model” for analyzing various different quantum aspects of black holes and gravity in general.

Metric satisfies

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{l^2} g_{\mu\nu}, \quad \Lambda = -\frac{1}{l^2}$$

1992. Banados, Teitelboim and Zanelli found that 2+1 gravity has a BH solution $\Longrightarrow$ SUPRISE!

BTZ black hole is similar to Schwarzschild and Kerr BH:

- event horizon (inner and outer for $J \neq 0$)
- final state of a collapse
- thermodynamic properties

but differs from them in:

- asymptotically anti-de Sitter (rather then flat)
- no singularity in the origin
- no Newtonian limit

Why the surprise?

- Since in three dimensions the full curvature tensor is completely determined by Ricci tensor

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} - \frac{1}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R$$

  any solution of Einstein equation with(without) a cosmological constant Λ

$$R_{\mu\nu} = 2\Lambda g_{\mu\nu}$$

  has constant(zero) curvature.

BTZ black hole and the propagation of a scalar test field

Probing black holes (BH) with scalar fields reveals important information about thermodynamical properties of the system, such as black hole entropy and Hawking radiation.

In this approach BH geometry is an external background on which scalar field is analyzed. There are potential problems with such analysis:

- behavior of the scalar field at the event horizon may not be well defined
- free energy may diverge due to infinite numbers of modes contributing to it near horizon
- certain spacetimes are not globally hyperbolic and that leads to difficulties in predicting the field propagation

- In order to deal with the difficulties arising from the lack of global hyperbolicity, one demands that the solution vanishes at spatial infinity.
- This solution can be then analytically continued to the horizon $r = l\sqrt{M}$ (or $z = 0$), where it diverges.
- In order to regulate this divergence, one usually introduces a "brick-wall" parameter ϵ, and one requires that the function vanishes at $z = \epsilon$. This boundary condition leads to a well posed problem, and the corresponding thermodynamical quantities can then be evaluated (but they depend on ϵ).

KG equation in BTZ

The KG equation for a massless scalar field in the BTZ background is given by

$$\Box\phi = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi\right) = 0$$

After using separation of variables $\phi = e^{-iEt}e^{im\varphi}R(r)$ we obtain the following radial equation

$$r\left(M - \frac{r^2}{l^2}\right)\frac{\partial^2 R}{\partial r^2} + \left(M - \frac{3r^2}{l^2}\right)\frac{\partial R}{\partial r} + \left(\frac{m^2}{r} - E^2\frac{r}{\frac{r^2}{l^2} - M}\right)R = 0,$$

where $m \in \mathbb{Z}$ is the azimuthal quantum number.

using $z = 1 - \frac{Ml^2}{r^2}$ the radial equation is

$$z(1-z)\frac{\mathrm{d}^2 R}{\mathrm{d}z^2} + (1-z)\frac{\mathrm{d}R}{\mathrm{d}z} + \left(\frac{A}{z} + B\right) R = 0,$$

where the constants A and B are

$$A = \frac{E^2 l^2}{4M}, \quad B = -\frac{m^2}{4M}.$$

This equation has regular singular points at $z = 0$, 1 and ∞, which means that around this points we can find a solutions via Frobenious method. The general solution can be written as

$$R(z) = z^\alpha F(a, b, c, z)$$

where $F(a, b, c, z)$ is the hypergeometric function and $a = \alpha + i\sqrt{-B}$, $b = \alpha - i\sqrt{-B}$, $c = 2\alpha + 1$ and $\alpha = i\sqrt{A}$

- Since $c \notin \mathbb{Z}$ we have two linearly independent solutions on the horizon, $F(a, b, c; z)$ and $z^{1-c} F(a+1-c,\ b+1-c,\ 2-c; z)$. So, the full solution is

$$\phi(z) = C_1 z^{\alpha} F(a, b, c; z) + C_2 z^{-\alpha} F(a+1-c,\ b+1-c,\ 2-c; z)$$

- Now we are interested in the behavior for $z \longrightarrow 0$. Since we have $F(..., z = 0) = 1$

$$\lim_{z \to 0} \phi(z) \propto C_1 z^{\alpha} + C_2 z^{-\alpha}$$

- We are now interested in the situation where there are NO square integrable solutions with the imaginary eigenvalue, i.e. $E \to \pm i$ and look at

$$\mathcal{L}_{\pm}[\phi_{\pm}] = \pm i \phi_{\pm}$$

- the complete measure is given by $\sqrt{-g} d^3x \approx dz$.
- So we are looking for a condition on α such that

$$\lim_{z \to 0} |\phi_{\pm}(z)|^2$$

  is divergent more then z^{-1}.
- Since $\alpha = -\frac{l}{2\sqrt{M}} < 0$, for the $+i$-case the second term in is regular and the divergence can come only from the first term, giving

$$\lim_{z \to 0} |\phi_{+}(z)|^2 \propto z^{2\alpha}$$

  Therefore, there will be NO square integrable solutions if

$$2\alpha \leq -1$$

  which then gives a condition on the BTZ parameters

$$\sqrt{M} \leq l$$

- Analogously, one can prove that for the same condition eigenvalue equation

$$\mathcal{L}_{-}[\phi_{-}] = -i\phi_{-}$$

has no square integrable solutions.

- Therefore we obtained for the deficiency indicies

$$(n_{+}, n_{-}) = (0, 0)$$

rendering the operator $\mathcal{L}$ essentially self-adjoint, so that the propagation of scalar field is uniquely defined for all times and we conclude that the BTZ spacetime is quantum complete for the parameter

$$\sqrt{M} \leq l$$

Naked singularity

When $M < 0$ we have a naked singularity at the origin.

For $r \longrightarrow 0 \Rightarrow ds^2 \cong -dt^2 + dr^2 + \alpha^2 r^2 d\phi^2$, $\alpha^2 = -M$

Equation of motion and the solution are given by

$$R'' + \frac{1}{r}R' + \left(\pm i - \frac{n^2}{\alpha^2 r^2}\right) R = 0, \quad R(r) \propto AJ_{n/\alpha}(r) + BN_{n\alpha}(r) \tag{1}$$

R is square integrable $\Rightarrow$ QM singular!

For the Dirac probe it becomes NON-singular!

J. P. M. Pitelli and P. S. Letelier, "Quantum singularities in the BTZ spacetime," Phys. Rev. D **77** (2008) 124030 [arXiv:0805.3926 [gr-qc]].

Physical motivation for NCG

- DFR, Commun. Math. Phys. 172: 187-220 (1995)

  „...**A sufficient condition for preventing gravitational collapse can be expressed as an uncertainty relation for the coordinates. This relation can in turn be derived from a commutation relation for the coordinates.**"

$$\Delta x_\mu \Delta x_\nu > l^2_{\text{Planck}}$$

$$x_\mu \to \hat{x}_\mu \Rightarrow [\hat{x}_\mu, \hat{x}_\nu] \neq 0$$

- Certain low energy limits of string theory (Moyal space) and Loop quantum gravity (κ-deformed) lead to NCFT
- **Not only for Planck scale physics** $\longrightarrow$ Almost-commutative manifolds: reformulation of gauge theories and the "mathematical" origin of Higgs mechanism and Standard model (Dubois-Violette, Kerner, Madore, Connes,...)

κ-Minkowski space

κ-Minkowski space is a Lie-algebraic deformation of the usual Minkowski space, where $\kappa \propto \frac{1}{a_0}$ is the deformation parameter, usually interpreted as the Planck mass or the new scale of quantum gravity

$$[\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{x}_0, \hat{x}_i] = ia_0\hat{x}_i.$$

Or more generally, we can introduce a deformation vector a_μ such that

$$[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu\hat{x}_\nu - a_\nu\hat{x}_\mu).$$

NC BTZ via NC duality

- In a previous set of works
  K. S. Gupta, E. Harikumar, T. Juric, S. Meljanac and A. Samsarov, "Effects of Noncommutativity on the Black Hole Entropy," Adv. High Energy Phys. **2014** (2014) 139172, [arXiv:1312.5100 [hep-th]]
  K. S. Gupta, E. Harikumar, T. Juric, S. Meljanac and A. Samsarov, "Noncommutative scalar quasinormal modes and quantization of entropy of a BTZ black hole," JHEP **1509** (2015) 025, [arXiv:1505.04068 [hep-th]]
  T. Juric and A. Samsarov, "Entanglement entropy renormalization for the noncommutative scalar field coupled to classical BTZ geometry," Phys. Rev. D **93** (2016) no.10, 104033 [arXiv:1602.01488 [hep-th]]
  K. S. Gupta, T. Juric and A. Samsarov, "Noncommutative duality and fermionic quasinormal modes of the BTZ black hole," JHEP **1706** (2017) 107, [arXiv:1703.00514 [hep-th]].
- It is established that probing a spinless BTZ black hole with a κ-Minkowski scalar field is equivalent to probing a spinning BTZ black hole with a commutative scalar field
- The effective spin of this dual BTZ black hole was obtained from the corresponding black hole entropy , and it was shown that it depends on the NC parameter.
- Moreover, it captures the backreaction of the NC scalar field on the BTZ spacetime.

NC KG equation

$$z(1-z)\frac{\mathrm{d}^2R}{\mathrm{d}z^2}+(1-z)\frac{\mathrm{d}R}{\mathrm{d}z}+\left(\frac{A}{z}+B\right)R=0,$$

where

$$A=\frac{l^4}{4(r_+^2-r_-^2)^2}\left(Er_+-\frac{m}{l}r_-\right)^2$$
$$B=\frac{-l^4}{4(r_+^2-r_-^2)^2}\left(Er_--\frac{m}{l}r_+\right)^2$$

where $(J^d(a))^2=\Lambda_{NC}\frac{64}{3}\pi\frac{\zeta(2)}{\zeta(3)}lM^{5/2}+O(a^2)$

- Repeating the same procedure as before, we get that the leading term of interest is

$$\lim_{z \to 0} |R(z)|^2 \propto z^{2\Re(\alpha)}$$

which now has to be more divergent than z^{-1} leadind to the following condition

$$\Re(\alpha) \leq -\frac{1}{2},$$

where $\Re(\alpha)$ stands for the real part of α.
- under this condition, the operator $\mathcal{L}$ is essentially self-adjoint and this ensures the unique time evolution for the scalar field in the NC setting.

Conclusion

- Evaluating this condition for the physical parameters in the NC setting we obtain

$$\sqrt{M} \leq l \left(1 + \Lambda_{NC} \frac{\zeta(2)}{\zeta(3)} \frac{56\pi}{3}\right)$$

- Comparing with the commutative case we see that the upper bound on the range of the BTZ parameters is enlarged, that is noncommutativity "smeared out" the singularity by allowing quantum completeness for a wider range of BTZ parameters.

- the naked singularity situation does not get improved for the scalar probe in the NC setting

Future prospective

- Further examine NC corrections to equations of motion in different backgrounds (hopefully in 4-D)
- Take that the very existence of the probe affects the spacetime and investigate the backreaction

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

via $T_{\mu\nu} = < T^{NC}_{\mu\nu} >$

motivated by

M. Casals, A. Fabbri, C. Martínez and J. Zanelli, "Quantum dress for a naked singularity," Phys. Lett. B **760**, 244 (2016) [arXiv:1605.06078 [hep-th]]

M. Casals, A. Fabbri, C. Martínez and J. Zanelli, "Quantum Backreaction on Three-Dimensional Black Holes and Naked Singularities," Phys. Rev. Lett. **118**, no. 13, 131102 (2017) [arXiv:1608.05366 [gr-qc]].

Thank you for your attention :)