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# Noncommutative Quasi-normal Modes & Holography





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# References

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# Plan

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- Introduction
- Scalar NC QNM's
- NC duality
- Fermionic NC QNM's
- NC holography
- Black hole entropy
- Concluding remarks

# Introduction

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- How to describe quantum structure of space-time? In particular:
  - Can space-time events be observed with arbitrary precision ?
  - Is there a fundamental and elementary length scale in nature ?
- These issues are related to the **quantum structure of space-time** relevant at the Planck scale.
- **Noncommutative Geometry** is one of the candidates for describing physics at that regime.

# Space-time UR

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## Heisenberg's Principle

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$\implies$  Space-time uncertainty relations

## Einstein's Theory

- Measuring a space-time coordinate with an accuracy  $\delta$  causes an uncertainty in the momentum  $\sim \frac{1}{\delta}$ .
- Neglecting rest mass, an energy of the order  $\frac{1}{\delta}$  is transmitted to the system and concentrated for some time in the localization region. The associated energy-momentum tensor generates a gravitational field.
- The smaller the uncertainties in the measurement of coordinates, the stronger will be the gravitational field generated by the measurement.

## Space-time UR

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- To probe physics at Planck Scale  $l_p$ , the Compton wavelength  $\frac{1}{M}$  of the probe must be less than  $l_p$ , hence  $M > \frac{1}{l_p}$ , i.e. Planck mass.
- When this field becomes so strong as to prevent light or other signals from leaving the region in question, an operational meaning can no longer be attached to the localization.
- Similarly, observations of very short time scales also require very high energies. Such observations can also limit spatial resolutions leading to a relation of the form

$$\Delta t \Delta x \geq L^2, \quad L = \text{fundamental length}$$

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## Space-time UR

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- Based on these arguments, Doplicher, Fredenhagen and Roberts (1994) arrived at uncertainty relations between the coordinates, which they showed could be deduced from a commutation relation of the type

$$[q_\mu, q_\nu] = iQ_{\mu\nu}$$

where  $q_\mu$  are self-adjoint coordinate operators,  $\mu, \nu$  run over space-time coordinates and  $Q_{\mu\nu}$  is an antisymmetric tensor, with the simplest possibility that it commutes with the coordinate operators.

- **NC geometry as a model of a quantum space-time.**

# Introduction

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- Noncommutative geometry is a beautiful idea by itself, supported by string theory, certain formulations of quantum gravity, DFT and so on
- What can be observational consequences of this framework?
- We will discuss a toy model which will illustrate the possibility of such consequences
- Our proposal is based on properties of gravitational waves



# Introduction

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- The first direct detection of gravitational waves from black hole mergers took place in 2015.
- The frequencies associated with the gravitational waves arising from an astrophysical event approximately 1.4 billion years ago were detected on earth.
- This discovery has ushered in a new field of gravitational wave astronomy.
- It is now possible to think of the of detection of gravitational waves from the primordial sources.
- These waves can in principle carry signatures of quantum structure of space-time.
- It is thus important to ask how quantum structure of space-time can affect the gravitational wave spectrum.

# Model

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To study gravitational waves in NC spacetime, we have three possibilities :

- NC geometry probed by a NC field
- NC geometry probed by commutative field
- Commutative geometry probed by a NC field

We have chosen the **third possibility** for our analysis.

This is in the similar spirit of a given classical geometry being probed by different type of fields to extract physical information.

# Model

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- We consider the action

$$\mathcal{S} = \int d^4x \sqrt{-g} g^{\mu\nu} (\partial_\mu \phi \star \partial_\nu \phi)$$

- The noncommutativity is chosen to be given by the  $\kappa$ -deformed algebra

$$[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu)$$

where we shall choose  $a_0 = \frac{1}{\kappa} \equiv a$  and  $a_i = 0$ .

- The  $\kappa$ -deformed algebra naturally appears in the NC description of a large class of black holes and in certain NC versions of cosmology. This motivates our choice.
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# Model

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- Operators  $\hat{x}_\mu$  can be realized in terms of the operators  $x_\mu$  and  $p_\mu (= i\partial_\mu)$  defined as

$$\hat{x}_\mu = x_\alpha \varphi^\alpha_\mu(p)$$

- From now on, we work upto the first order in the deformation parameter  $\alpha$ . Demanding consistency of the realization with the algebra gives

$$\varphi^\alpha_\mu = \delta^\alpha_\mu [1 + \alpha(a \cdot p)] + \beta a^\alpha p_\mu + \gamma p^\alpha a_\mu, \quad \alpha, \beta, \gamma \in \mathbb{R}, \quad \gamma - \alpha = 1$$

- Upto the first order, the action is given by

$$\mathcal{S} = \mathcal{S}_0 + \int d^4x \left( \mathcal{A}_{\alpha\beta\gamma\delta} \frac{\partial^2 \phi}{\partial x_\alpha \partial x_\beta} \frac{\partial^2 \phi}{\partial x_\gamma \partial x_\delta} \right),$$

where  $\mathcal{S}_0$  is the commutative action and

$$\mathcal{A}_{\alpha\beta\gamma\delta} = i\sqrt{-g} g_{\beta\delta} (\alpha x_\alpha a_\gamma + \beta(a \cdot x)\eta_{\alpha\gamma} + \gamma a_\alpha x_\gamma)$$

# Model

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- We shall next find the equation of motion to the first order in  $a$ . This is in general very complicated and to simplify it, we use the long wavelength approximation and keep terms only upto the lowest order in derivatives.
- We choose the classical geometry to be given by a massive spinless BTZ black hole. This is a simple background for which the quasi-normal modes can be studied analytically in the commutative case. This motivates the choice of the background.

## EOM- Scalar

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The massive spinless BTZ black hole is described by the metric

$$g_{\mu\nu} = \begin{pmatrix} \frac{r^2}{l^2} - M & 0 & 0 & 0 \\ 0 & -\frac{1}{\frac{r^2}{l^2} - M} & 0 & 0 \\ 0 & 0 & 0 & -r^2 \end{pmatrix},$$

We start with a massive spinless BTZ black hole and a  $\kappa$ -type NC scalar field. The equations of motion are derived from

$$\hat{\mathcal{S}} = \int d^4x \sqrt{-g} g^{\mu\nu} (\partial_\mu \phi \star \partial_\nu \phi).$$

where the NC star product has been defined before.

## EOM - Scalar

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Using the decomposition

$$\phi(r, \theta, t) = R(r)e^{-i\omega t}e^{im\theta}$$

the radial equation of motion upto first order in the NC parameter is

$$r \left( 8GM - \frac{r^2}{l^2} \right) \frac{\partial^2 R}{\partial r^2} + \left( 8GM - \frac{3r^2}{l^2} \right) \frac{\partial R}{\partial r} + \left( \frac{m^2}{r} - \omega^2 \frac{r}{\frac{r^2}{l^2} - 8GM} - a\beta\omega \frac{8r}{l^2} \frac{\frac{3r^2}{2l^2} - 8GM}{\frac{r^2}{l^2} - 8GM} \right) R = 0$$

## EOM - Scalar

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Using

$$z = 1 - \frac{Ml^2}{r^2},$$

we get

$$z(1-z)\frac{d^2 R}{dz^2} + (1-z)\frac{dR}{dz} + \left(\frac{A}{z} + B + \frac{C}{1-z}\right)R = 0,$$

$$A = \frac{\omega^2 l^2}{4M} + a\beta\omega, \quad B = -\frac{m^2}{4M}, \quad C = 3a\beta\omega.$$

These equations have very special features and we shall discuss those shortly.



## QNM - Scalar

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Using the ansatz

$$R(z) = z^{\lambda_1} (1 - z)^{\lambda_2} F(a, b, c, z)$$

we get

$$z(1 - z) \frac{d^2 F}{dz^2} + [c - (1 + a + b)z] \frac{dF}{dz} - abF = 0.$$

where

$$a = \lambda_1 + \lambda_2 + i\sqrt{-B}, \quad b = \lambda_1 + \lambda_2 - i\sqrt{-B} \quad c = 2\lambda_1 + 1$$

$$\lambda_1 = -i\sqrt{A} \quad \lambda_2 = \frac{1}{2}(1 - \sqrt{1 - 4C})$$

# QNM - Scalar

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The quasinormal modes are defined as solutions which are purely ingoing at the horizon, and which vanish at infinity. We have two linearly independent solutions  $F(a, b, c, z)$  and  $z^{1-c}F(a - c + 1, b - c + 1, 2 - c, z)$  near the horizon  $z = 0$ . Thus, the solution which has ingoing flux at the horizon is given by

$$R(z) = z^{\lambda_1} (1 - z)^{\lambda_2} F(a, b, c, z)$$

This is valid only in some neighborhood of the horizon, for the infinity,  $z = 1$ , we use analytic continuation

$$R(z) = z^{\lambda_1} (1 - z)^{\lambda_2 + c - a - b} \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} F(c - a, c - b, c - a - b + 1, 1 - z) \\ + z^{\lambda_1} (1 - z)^{\lambda_2} \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} F(a, b, a + b - c + 1, 1 - z),$$

Using the QNM boundary conditions we get

$$c - a = -n, \quad \text{or} \quad c - b = -n,$$

and  $n = 0, 1, 2, \dots$ . These conditions determine the frequencies of the quasinormal modes.

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## QNM - Scalar

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- The NC QNM frequencies are given by

$$\omega_{L,R} = \pm \frac{m}{l} + a\beta \frac{2M}{l^2} (6n + 5) - 2i \left[ \frac{\sqrt{M}}{l} (n + 1) \mp 3a\beta \frac{m}{l^2} \sqrt{M} \right]$$

$$n = 0, 1, 2, \dots$$

- The gravity waves in principle provide an opportunity to observe quantum gravity effects as described by NC physics.

# NC Duality

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- We started with a massive spinless BTZ black hole probed by a massless NC scalar field.
- The equation of motion that we got corresponds to that of a massive spinning BTZ black hole probed by a massive commutative scalar field.
- Within our scheme of approximation, this is a new kind of duality.
- Thus we have
$$M^f = M^f(a, M), \quad J^f = J^f(a, M), \quad \mu^f = \mu^f(a, M)$$
where  $M^f$  and  $J^f$  are the mass and spin of the dual black hole and  $\mu^f$  is the mass of the scalar field.

# NC Duality

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- Now we determine the parameters of the dual black hole?
- For that, we first calculate the entropy of the original black hole with mass  $M$  using the brick wall method and get

$$S^{NC} = \frac{2\pi l\sqrt{M}}{4G} \left( 1 + a\beta\sqrt{M} \frac{8\pi\zeta(2)}{3l\zeta(3)} \right)$$

- For the dual spinning black hole

$$S^d = \frac{2\pi r_+}{4G}, \quad r_+ = \frac{l\sqrt{M}}{\sqrt{2}} \sqrt{1 + \sqrt{1 - \frac{(J^d)^2}{M^2 l^2}}}$$

- We now demand  $S^{NC} = S^d$

# NC Duality

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- This leads to

$$(J^d(a))^2 = \lambda \frac{64}{3} \pi \frac{\zeta(2)}{\zeta(3)} l M^{5/2} + O(a^2) \quad \lambda = -a\beta$$

This implies that  $\beta < 0$ .

- Thus the dual black hole has mass  $M$  and spin  $J^d$ .
- The scalar field probe also picks up an effective mass proportional to  $-a\beta$ . This is consistent with the restriction that  $\beta < 0$ .
- We could have taken  $M$  to depend on the parameter  $a$ . In that case we can show that  $M$  picks up a correction to higher order than that for  $J^d$ .

## QNM - Fermionic

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- We would like to probe the black hole with a fermionic field
- This is a very involved procedure. To simplify the method, we use the dual black hole picture discussed above.
- Instead of probing a massless spinning BTZ with a NC fermionic probe, we probe the dual black hole with a commutative fermionic probe.
- We ensure that  $D^2 = \square_g$ , where  $\square_g$  corresponds to the KG operator for the dual black hole with mass  $M$  and  $J \propto \sqrt{a\beta}$ .

# QNM - Fermionic

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We start with the metric

$$ds^2 = -\left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}\right) dt^2 + \frac{dr^2}{\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}} + r^2 \left(d\phi - \frac{J}{2r^2} dt\right)^2.$$

In terms of coordinates

$$x^+ = \frac{1}{l} r_+ t - r_- \phi, \quad x^- = r_+ \phi - \frac{1}{l} r_- t \quad \text{and} \quad \tanh \rho = \sqrt{\frac{r^2 - r_+^2}{r^2 - r_-^2}}$$

$$ds^2 = -\sinh^2 \rho (dx^+)^2 + l^2 d\rho^2 + \cosh^2 \rho (dx^-)^2.$$

We choose

$$\gamma^0 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



# QNM - Fermionic

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The Dirac equation is given by

$$\left[ \gamma^a e_a{}^\mu \left( \partial_\mu - \frac{i}{2} \omega_\mu{}^{cd} \Sigma_{cd} \right) + m \right] \Psi = 0$$

where  $e^0_{x_+} = \sinh \rho$ ,  $e^1_\rho = l$ ,  $e^2_{x_-} = \cosh \rho$  and others being 0 and

$$\omega_0{}^{01} \equiv \omega_{x_+}{}^{01} = -\omega_0{}^{10} = \frac{1}{l} \cosh \rho,$$

$$\omega_2{}^{12} \equiv \omega_{x_-}{}^{12} = -\omega_2{}^{21} = -\frac{1}{l} \sinh \rho,$$

This leads to the Dirac equation

$$\left[ \frac{1}{l} \gamma^1 \left( \frac{\partial}{\partial \rho} + \frac{\cosh \rho}{2 \sinh \rho} + \frac{\sinh \rho}{2 \cosh \rho} \right) + \gamma^0 \frac{1}{\sinh \rho} \frac{\partial}{\partial x^+} + \gamma^2 \frac{1}{\cosh \rho} \frac{\partial}{\partial x^-} + m \right] \Psi = 0$$

# QNM - Fermionic

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We choose the ansatz

$$\begin{aligned}\Psi &= \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_1(r) \\ \psi_2(r) \end{pmatrix} \exp \left[ -\frac{i}{l} (\omega t - j\phi) \right] = \begin{pmatrix} \psi_1(\rho) \\ \psi_2(\rho) \end{pmatrix} \exp [-i(k_+ x^+ + k_- x^-)] \\ &= \frac{1}{2} \begin{pmatrix} P(\rho) + Q(\rho) \\ P(\rho) - Q(\rho) \end{pmatrix} \exp [-i(k_+ x^+ + k_- x^-)],\end{aligned}$$

where  $\omega$  and  $j$  are respectively the energy and angular momentum of the spin 1/2 particle and they are related to  $k_+$  and  $k_-$  as

$$k_+ = \frac{l\omega r_+ - jr_-}{l(r_+^2 - r_-^2)}, \quad k_- = \frac{l\omega r_- - jr_+}{l(r_+^2 - r_-^2)}.$$

Now defining

$$P(\rho) = \sqrt{\frac{\cosh \rho + \sinh \rho}{\cosh \rho \sinh \rho}} P'(\rho), \quad Q(\rho) = \sqrt{\frac{\cosh \rho - \sinh \rho}{\cosh \rho \sinh \rho}} Q'(\rho), \quad z = \tanh^2 \rho$$

and further putting  $P' = \psi'_1 + \psi'_2$  and  $Q' = \psi'_1 - \psi'_2$

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# QNM - Fermionic

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We get

$$2\sqrt{z}(1-z)\frac{d}{dz}\psi'_1 + il\left(k_+\frac{1}{\sqrt{z}} + k_-\sqrt{z}\right)\psi'_1 + \left[il(k_+ + k_-) + lm + \frac{1}{2}\right]\psi'_2 = 0,$$
$$2\sqrt{z}(1-z)\frac{d}{dz}\psi'_2 - il\left(k_+\frac{1}{\sqrt{z}} + k_-\sqrt{z}\right)\psi'_2 - \left[il(k_+ + k_-) - lm - \frac{1}{2}\right]\psi'_1 = 0.$$

which can be combined to give

$$z(1-z)\frac{d^2}{dz^2}\psi'_1 + \frac{1-3z}{2}\frac{d}{dz}\psi'_1 + \frac{1}{4}\left[\frac{l^2k_+^2 - ilk_+}{z} + ilk_- - l^2k_-^2 - \frac{(lm + \frac{1}{2})^2}{1-z}\right]\psi'_1 = 0.$$

The solutions with purely ingoing flux at the horizon are given by

$$\psi'_1 = z^\alpha(1-z)^\beta F(a, b, c; z),$$

$$\psi'_2 = \left(\frac{a-c}{c}\right) z^{\alpha+\frac{1}{2}}(1-z)^\beta F(a, b+1, c+1; z),$$

# QNM - Fermionic

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where

$$\alpha = -\frac{ilk_+}{2}, \quad \beta = -\frac{1}{2} \left( lm + \frac{1}{2} \right), \quad c = 2\alpha + \frac{1}{2},$$

$$a = \alpha + \beta + \frac{ilk_-}{2} + \frac{1}{2} = \frac{l(k_+ - k_-)}{2i} + \beta + \frac{1}{2},$$

$$b = \alpha + \beta - \frac{ilk_-}{2} = \frac{l(k_+ + k_-)}{2i} + \beta.$$

QNM boundary conditions also require vanishing flux at infinity. The flux in the radial direction is given by

$$\mathcal{J}_\rho = \frac{l^2}{\sqrt{1-z}} \left[ \psi_1'^* \psi_2' + \psi_2'^* \psi_1' + \sqrt{z} \left( \psi_1'^* \psi_1' + \psi_2'^* \psi_2' \right) \right]$$

Demanding that the outgoing flux vanishes at infinity requires that

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = 0.$$

This requires either  $c - a = -n$  or  $c - b = -n$ , with  $n = 0, 1, 2, \dots$

## QNM - Fermionic

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These equations lead to the conditions

$$\omega_L = \frac{j}{l} - 2i \frac{r_+ - r_-}{l} \left( n + \frac{lm}{2} + \frac{1}{4} \right),$$
$$\omega_R = -\frac{j}{l} - 2i \frac{r_+ + r_-}{l} \left( n + \frac{lm}{2} + \frac{3}{4} \right).$$

The QNM frequencies upto the first order in the deformation parameter are given by

$$\omega_L = \frac{j}{l} - 2i\sqrt{M} \left( 1 - \frac{J(a)}{2lM} - \frac{1}{8} \frac{J^2(a)}{l^2 M^2} \right) \left( n + \frac{1}{4} + \frac{lm}{2} \right) + O(a^{3/2})$$
$$\omega_R = -\frac{j}{l} - 2i\sqrt{M} \left( 1 + \frac{J(a)}{2lM} - \frac{1}{8} \frac{J^2(a)}{l^2 M^2} \right) \left( n + \frac{3}{4} + \frac{lm}{2} \right) + O(a^{3/2})$$

# Holography

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- Holography and AdS/CFT duality are fundamental aspects in certain quantum theories of gravity, such as string theory.
- NC effects are also relevant at the Planck scale.
- Thus it is natural to investigate holography within the NC framework.
- One approach is through QNM's, which play an important role in holography, especially for the BTZ.
- Second approach is via Sullivan's theorem, which is relevant for the BTZ.

# Holography

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- For the BTZ black hole, the poles of the retarded Green's function in the boundary CFT are in exact correspondence with the QNM frequencies in the bulk
- This provides a demonstration for the AdS/CFT conjecture for the BTZ
- The NC duality allows us to discuss our problem in the framework of a commutative BTZ, for which the AdS/CFT conjecture holds
- We can thus predict that the poles of the retarded Green's function would pick up NC corrections as

$$2i\sqrt{M} \left( \frac{J(a)}{2lM} + \frac{J^2(a)}{8l^2 M^2} \right) - 2i\sqrt{M} \left( \frac{J(a)}{2lM} - \frac{J^2(a)}{8l^2 M^2} \right)$$

# Holography

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- Sullivan's theorem says that for a certain class of manifolds, there is a 1-1 correspondence of the hyperbolic structure as encoded in the metric and the conformal structure of the boundary.
- It has been shown to be valid for the BTZ.
- Sullivan's theorem implies that the monodromies of the solutions of the wave equation around the two horizons of the BTZ satisfy

$$\mathcal{M}(r_+) \mathcal{M}^+(r_-) = 1 \quad \mathcal{M}(r_+) \mathcal{M}^-(r_-) = 1$$



# Holography

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- These conditions lead to

$$\omega_L = \frac{j}{l} - 2i \frac{r_+ - r_-}{l} \left( n + \frac{1}{4} \right),$$
$$\omega_R = -\frac{j}{l} - 2i \frac{r_+ + r_-}{l} \left( n + \frac{3}{4} \right).$$

which are the same equations that we got by direct calculation restricted to the massless probe.

- These modes are obtained without using any condition at infinity and hence are also known as non-QNM's.
- They are obtained purely from holographic considerations as applied to the BTZ.

# Entropy

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- We have analyzed black hole entropy within our framework in two different ways.
- The first one involves the brick wall method introduced by 't Hooft.
- The second one involves the study of the area operator and the quantization of entropy.
- Our results lead to a NC correction to the usual BTZ entropy.
- Our analysis predicts a renormalization of the Newton's constant due to NC corrections.

# Entropy - Brick Wall Method

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In the WKB approximation, the  $r$ -dependent radial wavefunction has the form

$R(r) = e^{i \int k(r) dr}$  where

$$k^2(r, m, \omega) = -\frac{m^2}{r^2 \left( \frac{r^2}{l^2} - 8GM \right)} + \omega^2 \frac{1}{\left( \frac{r^2}{l^2} - 8GM \right)^2} + a\beta\omega \frac{8}{l^2} \frac{\frac{3r^2}{2l^2} - 8GM}{\left( \frac{r^2}{l^2} - 8GM \right)^2}$$

According to the semi-classical quantization rule, the radial wave number is quantized as

$$\pi n = \int_{r_++h}^L k(r, m, \omega) dr$$

where the quantum number  $n > 0$ ,  $m$  should be fixed such that  $k(r, m, \omega)$  is real and  $h$  is the brick wall cutoff (UV regulator) and  $L$  is the infrared regulator. The total number  $\nu$  of solutions with energy not exceeding  $\omega$  is given by

$$\nu = \sum_{-m_0}^{m_0} n = \int_{-m_0}^{m_0} dm n = \frac{1}{\pi} \int_{-m_0}^{m_0} dm \int_{r_++h}^L k(r, m, \omega) dr$$

# Entropy - Brick Wall Method

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The free energy at inverse temperature  $\beta_T$  of the black hole is

$$\begin{aligned} e^{-\beta_T F} &= \sum_{\nu} e^{-\beta_T E} = \prod_{\nu} \frac{1}{1 - e^{-\beta_T E}} \quad / \ln \\ \beta_T F &= \sum_{\nu} \ln \left( 1 - e^{-\beta_T E} \right) = \int d\nu \ln \left( 1 - e^{-\beta_T E} \right) \quad / \text{part. integ.} \\ &= - \int_0^{\infty} dE \frac{\beta_T \nu(E)}{e^{\beta_T E} - 1} \end{aligned}$$

where  $\beta_T = \frac{2\pi l^2}{r_+}$ . For this, we find the free energy  $F$  as

$$F = - \frac{1}{\pi} \int_0^{\infty} \frac{d\omega}{e^{\beta_T \omega} - 1} \int_{r_++h}^L dr \int_{-m_0}^{m_0} dm k(r, m, \omega)$$

Keeping the most divergent terms in  $h$ , we get

$$F = - \frac{l^{\frac{5}{2}}}{(8GM)^{\frac{1}{4}}} \frac{\zeta(3)}{\beta_T^3} \frac{1}{\sqrt{2h}} - 2a\beta \frac{(8GM)^{\frac{3}{4}} \sqrt{l} \zeta(2)}{\sqrt{2h} \beta_T^2}$$

# Entropy - Brick Wall Method

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Using  $S = \beta_T^2 \frac{\partial F}{\partial \beta_T}$ , we get

$$\begin{aligned} S &= 3 \frac{l^{\frac{5}{2}}}{(8GM)^{\frac{1}{4}}} \frac{\zeta(3)}{\beta_T^2} \frac{1}{\sqrt{2h}} + 4a\beta \frac{(8GM)^{\frac{3}{4}} \sqrt{l} \zeta(2)}{\sqrt{2h} \beta_T} \\ &= S_0 \left( 1 + \frac{4}{3} a\beta \frac{8GM \zeta(2)}{l^2 \zeta(3)} \beta_T \right) \end{aligned}$$

where  $S_0$  is the undeformed entropy for BTZ and

$$h = \frac{9G^2 \zeta^2(3) \sqrt{8GM}}{8l\pi^6} \quad (1)$$

The cutoff  $h$  is fixed by demanding that  $S_0 = \frac{A}{4G} = \frac{2\pi r_+}{4G}$ .

# Entropy - Renormalization of $G$

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Now writing  $S = \frac{A}{4G^*}$ , we find that

$$\frac{1}{G^*} = \frac{1}{G} \left( 1 + \frac{8}{3} \frac{a\beta\pi}{l} \frac{\zeta(2)}{\zeta(3)} \sqrt{8GM} \right)$$

This gives a renormalization of the Newton's constant due to NC effects

# Entropy - Quantization

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- Given a system with energy  $E$  and vibrational frequency  $\Delta\omega(E)$ , it can be shown that

$$\mathcal{I} = \int \frac{\delta E}{\Delta\omega(E)}$$

is an adiabatic invariant.

- By Bohr-Sommerfeld semi-classical quantization,

$$\mathcal{I} \approx n\hbar$$

- We identify the energy  $E$  with the black hole mass  $M$ .
- The frequency is identified with the absolute value of QNM frequency.

# Entropy - Quantization

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Using the formulae

$$\Delta M = \hbar \Delta \omega = \hbar (|\omega_{L,R}|_n - |\omega_{L,R}|_{n-1})$$

$$\Delta \omega = |\omega_{L,R}|_n - |\omega_{L,R}|_{n-1} = \frac{2\sqrt{M}}{l} \left( 1 \pm \frac{a\beta}{2l} \frac{m}{n(n+1)} \right)$$

$$\Delta \omega = \frac{2\sqrt{M}}{l} \left( 1 \pm \frac{a\beta}{2l} \frac{m}{n(n+1)} \right).$$

$$A = 2\pi r_+ = 2\pi l \sqrt{M},$$

$$\Delta A = 2\pi \hbar \left( 1 \pm \frac{a\beta}{2l} \frac{m}{n(n+1)} \right).$$

we can show that

$$\mathcal{I} = \int \frac{\delta M}{\Delta \omega} \approx l \sqrt{M} \left( 1 \mp \frac{a\beta}{2l} \frac{m}{n(n+1)} \right) = N\epsilon, \quad N \in \mathbb{N}$$



## Entropy - Quantization

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Using these equations we find

$$A = 2\pi\mathcal{I} \left( 1 \pm \frac{a\beta}{2l} \frac{m}{n(n+1)} \right) \Rightarrow A_N = 2\pi N\epsilon \left( 1 \pm \frac{a\beta}{2l} \frac{m}{n(n+1)} \right).$$

Using the Bekenstein-Hawking relation with a renormalized  $G$ , we have

$$S_N = \frac{A_N}{4G^*} = \frac{A_N}{4G} \left( 1 + a\beta \frac{8\pi}{3l} \frac{\zeta(2)}{\zeta(3)} \sqrt{8GM} \right)$$

we are led to a quantized entropy

$$S_N = N\epsilon \frac{\pi}{2G} \left( 1 \pm \frac{a\beta}{2l} \frac{m}{n(n+1)} + a\beta \frac{8\pi}{3l} \frac{\zeta(2)}{\zeta(3)} \sqrt{8GM} \right)$$

## Concluding Remarks

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- We have investigated a toy model which illustrates the possibility of capturing Planck scale effects through gravitational waves.
- The QNM's explicitly depend on the NC parameter, which can be used put constraints on such parameters.
- Upto the first order in the deformation parameter, a new kind of black hole duality in  $AdS_3$  has been found.
- We have a prediction for the retarded pole of Green's function in a NC boundary field theory.
- Analysis of various other physical effects and more realistic backgrounds are open areas.