Conference on Symmetries, Geometry and Quantum Gravity
18-22 June 2018, Primošten, Croatia

# Nonassociative differential geometry and gravity

Marija Dimitrijević Ćirić

University of Belgrade, Faculty of Physics,
Belgrade, Serbia

with: P. Aschieri, R. Szabo;

arXiv: 1710.11467

# NC/NA geometry and gravity

Early Universe, singularities of BHs $\Rightarrow$ QG $\Rightarrow$ Quantum space-time
NC/NA space-time $\Longrightarrow$ Gravity on NC/NA spaces.

General Relativity (GR) is based on the diffeomorphism symmetry.
This concept (space-time symmetry) is difficult to generalize to NC/NA spaces. Different approaches:

**NC spectral geometry** [Chamseddine, Connes, Marcolli '07; Chamseddine, Connes, Mukhanov '14].

**Emergent gravity** [Steinacker '10, '16].

**Frame formalism, operator description** [Burić, Madore '14; Fritz, Majid '16].

**Twist approach** [Wess et al. '05, '06; Ohl, Schenckel '09; Castellani, Aschieri '09; Aschieri, Schenkel '14].

**NC gravity as a gauge theory of Lorentz/Poincaré group** [Chamseddine '01,'04, Cardela, Zanon '03, Aschieri, Castellani '09,'12; Dobrski '16].

# Overview

# NA gravity: Motivation

Do closed strings provide a framework for quantum gravity?

In particular: closed strings propagating in a locally non-geometric constant $R$-flux background [Munich group '11].

The low energy limit: Nonassociative gravity on space-time.

Twist deformation: a well defined way to deform symmetries, in particular diffeomorphism symmetry, and the corresponding differential geometry [Wess et al. '06; Aschieri et al. '08,'09,...].

$R$-flux induced cochain twist: introduced in [Mylonas, Schupp, Szabo '14; Aschieri, Szabo '15].

# NA gravity: General

NA gravity is based on:

-locally non-geometric constant $R$-flux.
-twist $\mathcal{F}$ and associator $\Phi$ with

$$\Phi\,(\mathcal{F}\otimes 1)\,(\Delta\otimes\mathrm{id})\mathcal{F} = (1\otimes\mathcal{F})\,(\mathrm{id}\otimes\Delta)\mathcal{F}\ .$$

-equivariance (covariance) under the twisted diffeomorphisms (quasi-Hopf algebra of twisted diffeomorphisms).
-twisted differential geometry in phase space. In particular: connection, curvature, torsion.
-NA Levi-Civita connection and projection of phase space Einstein equations to space-time.

Our goals:

-consistently construct NA deformation of GR: NA Einstein equations, NA Einstein-Hilbert action; investigate phenomenological consequences.
-understand symmetries of the obtained NA gravity.

# NA differntial geometry: Review of twist deformation

Symmetry algebra $g$ and the universal covering algebra $Ug$.
A well defined way of deforming symmetries: the twist formalism.
Twist $\mathcal{F}$ (introduced by Drinfel'd in 1983-1985) is:
-an invertible element of $Ug \otimes Ug$
-fulfills the 2-cocycle condition (ensures the associativity of the $\star$-product).

$$\mathcal{F} \otimes 1(\Delta \otimes \mathrm{id})\mathcal{F} = 1 \otimes \mathcal{F}(\mathrm{id} \otimes \Delta)\mathcal{F}. \tag{2.1}$$

-additionaly: $\mathcal{F} = 1 \otimes 1 + \mathcal{O}(h)$; $h$-deformation parameter.

Moyal-Weyl deformation: $\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}$

NC field theory [Szabo '03], NC Standard model [Wess et al. '01,...'06], NC gravity [Chamseddine '01,'04; Cardela, Zanon '03; Wess et al. '05...].

Abelian twist: $\mathcal{F} = e^{-\frac{i}{2}\theta^{ab}X_a \otimes X_b}$,

where $X_a = X_a^\mu(x)\partial_\mu$, $[X^a, X^b] = 0$ and $\theta^{ab} = const.$. NC gravity [Aschieri, Castellani '09, ...'14].

# NA differntial geometry: $R$-flux induced cochain twist

Phase space $\mathcal{M}$: $x^A = (x^\mu, \tilde{x}_\mu = p_\mu)$, $\partial_A = \left(\partial_\mu, \tilde{\partial}^\mu = \frac{\partial}{\partial p_\mu}\right)$.
$2d$ dimensional, $A = 1, \ldots 2d$.

The twist $\mathcal{F}$:

$$\mathcal{F} = \exp\left(-\tfrac{\mathrm{i}\,\hbar}{2}\,(\partial_\mu \otimes \tilde{\partial}^\mu - \tilde{\partial}^\mu \otimes \partial_\mu) - \tfrac{\mathrm{i}\,\kappa}{2}\,R^{\mu\nu\rho}\,(p_\nu\,\partial_\rho \otimes \partial_\mu - \partial_\mu \otimes p_\nu\,\partial_\rho)\right)\,, \quad (2.2)$$

with $R^{\mu\nu\rho}$ totally antisymmetric and constant, $\kappa := \frac{\ell_s^3}{6\hbar}$.
Does not fulfill the 2-cocycle condition

$$\Phi\,(\mathcal{F} \otimes 1)\,(\Delta \otimes \mathrm{id})\mathcal{F} = (1 \otimes \mathcal{F})\,(\mathrm{id} \otimes \Delta)\mathcal{F}\;. \quad (2.3)$$

The associator $\Phi$:

$$\Phi = \exp\left(\hbar\kappa\,R^{\mu\nu\rho}\,\partial_\mu \otimes \partial_\nu \otimes \partial_\rho\right) =: \phi_1 \otimes \phi_2 \otimes \phi_3 = 1 \otimes 1 \otimes 1 + O(\hbar\kappa). \quad (2.4)$$

Notation: $\mathcal{F} = \mathrm{f}^\alpha \otimes \mathrm{f}_\alpha$, $\mathcal{F}^{-1} = \bar{\mathrm{f}}^\alpha \otimes \bar{\mathrm{f}}_\alpha$, $\Phi^{-1} =: \bar{\phi}_1 \otimes \bar{\phi}_2 \otimes \bar{\phi}_3$,

$$\mathcal{R} = \mathcal{F}^{-2} =: \mathrm{R}^{\,\alpha} \otimes \mathrm{R}_{\,\alpha},\ \mathcal{R}^{-1} = \mathcal{F}^2 =: \overline{\mathrm{R}}^{\,\alpha} \otimes \overline{\mathrm{R}}_{\,\alpha}.$$

Hopf aglebra of diffeomorphisms $U\mathrm{Vec}(\mathcal{M})$:

$$[u,v] = (u^B\partial_B v^A - v^B\partial_B u^A)\partial_A,$$
$$\Delta(u) = 1\otimes u + u\otimes 1,$$
$$\epsilon(u) = 0, S(u) = -u.$$

Quasi-Hopf algebra of infinitesimal diffeomorphisms $U\mathrm{Vec}^{\mathcal{F}}(\mathcal{M})$:

-algebra structure does not change

-coproduct is deformed: $\Delta^{\mathcal{F}}\xi = \mathcal{F}\,\Delta\,\mathcal{F}^{-1}$

-counit and antipod do not change: $\epsilon^{\mathcal{F}} = \epsilon, S^{\mathcal{F}} = S$.

On basis vectors:

$$\Delta_{\mathcal{F}}(\partial_\mu) = 1\otimes\partial_\mu + \partial_\mu\otimes 1\ ,$$
$$\Delta_{\mathcal{F}}(\tilde\partial^\mu) = 1\otimes\tilde\partial^\mu + \tilde\partial^\mu\otimes 1 + \mathrm{i}\,\kappa\,R^{\mu\nu\rho}\,\partial_\nu\otimes\partial_\rho\ .$$

# NA differential geometry: NA tensor calculus

Guiding principle: Differential geometry on $\mathcal{M}$ is covatiant under $U\mathrm{Vec}(\mathcal{M})$.
NA differential geometry on $\mathcal{M}$ should be covariant under $U\mathrm{Vec}^{\mathcal{F}}(\mathcal{M})$.

In pactice: $U\mathrm{Vec}(\mathcal{M})$-module algebra $\mathcal{A}$ (functions, forms, tensors) and $a, b \in \mathcal{A}$, $u \in \mathrm{Vec}(\mathcal{M})$

$$u(ab) = u(a)b + au(b), \qquad \text{Lie derivative, coproduct.}$$

The twist: $U\mathrm{Vec}(\mathcal{M}) \to U\mathrm{Vec}^{\mathcal{F}}(\mathcal{M})$ and $\mathcal{A} \to \mathcal{A}_\star$ with $ab \to a \star b = \bar{\mathrm{f}}^{\alpha}(a) \cdot \bar{\mathrm{f}}_{\alpha}(b)$. Then $\mathcal{A}_\star$ is a $U\mathrm{Vec}^{\mathcal{F}}(\mathcal{M})$-module algebra:

$$\xi(a \star b) = \xi_{(1)}(a) \star \xi_{(2)}(b),$$

for $\xi \in U\mathrm{Vec}^{\mathcal{F}}(\mathcal{M})$ and using the twisted coproduct $\Delta^{\mathcal{F}}\xi$.

Commutativity: $a \star b = \overline{\mathrm{f}}^{\,\alpha}(a) \cdot \overline{\mathrm{f}}_{\,\alpha}(b) = \overline{\mathrm{R}}^{\,\alpha}(b) \star \overline{\mathrm{R}}_{\,\alpha}(a) =: {}^{\alpha}b \star_{\alpha} a$

Associativity: $(a \star b) \star c = {}^{\phi_1}a \star ({}^{\phi_2}b \star {}^{\phi_3}c)$.

Functions: $C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})_\star$

$$\begin{aligned} f \star g &= \overline{\mathrm{f}}^{\,\alpha}(f) \cdot \overline{\mathrm{f}}_{\,\alpha}(g) \\ &= f \cdot g + \tfrac{\mathrm{i}\,\hbar}{2}\left(\partial_\mu f \cdot \tilde{\partial}^\mu g - \tilde{\partial}^\mu f \cdot \partial_\mu g\right) + \mathrm{i}\,\kappa\, R^{\mu\nu\rho}\, p_\nu\, \partial_\rho f \cdot \partial_\mu g + \cdots\ , \end{aligned} \tag{2.5}$$

$$[x^\mu \stackrel{\star}{,} x^\nu] = 2\,\mathrm{i}\,\kappa\, R^{\mu\nu\rho}\, p_\rho,\ [x^\mu \stackrel{\star}{,} p_\nu] = \mathrm{i}\,\hbar\,\delta^\mu{}_\nu,\ [p_\mu \stackrel{\star}{,} p_\nu] = 0,$$

$$[x^\mu \stackrel{\star}{,} x^\nu \stackrel{\star}{,} x^\rho] = \ell_s^3\, R^{\mu\nu\rho}.$$

Forms: $\Omega^\sharp(\mathcal{M}) \to \Omega^\sharp(\mathcal{M})_\star$

$$\begin{aligned} \omega \wedge_\star \eta &= \overline{\mathrm{f}}^{\,\alpha}(\omega) \wedge \overline{\mathrm{f}}_{\,\alpha}(\eta), \\ f \star \mathrm{d}x^A &= \mathrm{d}x^C \star \left(\delta^A{}_C\, f - \mathrm{i}\,\kappa\, \mathscr{R}^{AB}{}_C\, \partial_B f\right)\ , \end{aligned} \tag{2.6}$$

with non-vanishing components $\mathscr{R}^{x^\mu, x^\nu}{}_{\tilde{x}_\rho} = R^{\mu\nu\rho}$. Basis 1-forms

$$\begin{aligned} (\mathrm{d}x^A \wedge_\star \mathrm{d}x^B) \wedge_\star \mathrm{d}x^C &= {}^{\phi_1}(\mathrm{d}x^A) \wedge_\star \left({}^{\phi_2}(\mathrm{d}x^B) \wedge_\star {}^{\phi_3}(\mathrm{d}x^C)\right) \\ &= \mathrm{d}x^A \wedge_\star (\mathrm{d}x^B \wedge_\star \mathrm{d}x^C) = \mathrm{d}x^A \wedge \mathrm{d}x^B \wedge \mathrm{d}x^C. \end{aligned}$$

# NA tensor calculus: duality, derivation, Lie derivative

Exterior derivative $\mathrm{d}$: $\mathrm{d}^2 = 0$ and the undeformed Leibniz rule

$$\mathrm{d}(\omega \wedge_\star \eta) = \mathrm{d}\omega \wedge_\star \eta + (-1)^{|\omega|}\, \omega \wedge_\star \mathrm{d}\eta. \tag{2.7}$$

Duality, $\star$-pairing:

$$\langle\, \omega\,,\, u\, \rangle_\star = \left\langle\, \overline{\mathrm{f}}^{\,\alpha}(\omega)\,,\, \overline{\mathrm{f}}_{\,\alpha}(u)\, \right\rangle\,. \tag{2.8}$$

$\star$ Lie drivative:

$$\mathcal{L}^\star_u(T) = \mathcal{L}_{\overline{\mathrm{f}}^{\,\alpha}(u)}(\overline{\mathrm{f}}_{\,\alpha}(T)), \tag{2.9}$$

$$[\mathcal{L}^\star_u, \mathcal{L}^\star_v]_\bullet = \mathcal{L}^\star_{[u,v]_\star},$$

$$\mathcal{L}^\star_u(\omega \wedge_\star \eta) = \mathcal{L}^\star_{\bar\phi_1 u}({}^{\bar\phi_2}\omega) \wedge_\star {}^{\bar\phi_3}\eta + {}^{\alpha}({}^{\bar\phi_1\,\bar\varphi_1}\omega) \wedge_\star \mathcal{L}^\star_{\alpha({}^{\bar\phi_2\,\bar\varphi_2}u)}({}^{\bar\phi_3\,\bar\varphi_3}\eta),$$

with $[u,v]_\star = \left[\,\overline{\mathrm{f}}^{\,\alpha}(u), \overline{\mathrm{f}}_{\,\alpha}(v)\right]$. Relation of $\mathcal{L}^\star_u$ with diffeomorphism symmetry in space-time needs to be understood.

# NA differential geometry: NA connection

$\star$-connection:

$$\begin{aligned} \nabla^\star \ : \ \mathrm{Vec}_\star &\longrightarrow \mathrm{Vec}_\star \otimes_\star \Omega^1_\star \\ u &\longmapsto \nabla^\star u \ , \qquad (2.10) \\ \nabla^\star(u \star f) &= \left(^{\bar\phi_1}\nabla^\star(^{\bar\phi_2}u)\right) \star {}^{\bar\phi_3} f + u \otimes_\star \mathrm{d}f, \qquad (2.11) \end{aligned}$$

the right Leibniz rule, for $u \in \mathrm{Vec}_\star$ and $f \in A_\star$. In particular:

$$\nabla^\star \partial_A =: \partial_B \otimes_\star \Gamma^B_A =: \partial_B \otimes_\star (\Gamma^B_{AC} \star \mathrm{d}x^C) \ . \qquad (2.12)$$
$$\mathrm{d}_{\nabla^\star}(\partial_A \otimes_\star \omega^A) = \partial_A \otimes_\star (\mathrm{d}\omega^A + \Gamma^A_B \wedge_\star \omega^B),$$

for $\omega^A \in \Omega^\sharp_\star$.

Connection on forms, dual connection ${}^\star\nabla\omega$:

$$\begin{aligned} {}^\star\nabla \ : \ \Omega^1_\star &\longrightarrow \Omega^1_\star \otimes_\star \Omega^1_\star, \\ \omega &\longmapsto {}^\star\nabla\omega, \\ {}^\star\nabla(f \star \omega) &= {}^{\phi_1}f \star \left({}^{\phi_3\,\star}\nabla({}^{\phi_2}\omega)\right) + \mathrm{d}f \otimes_\star \omega. \qquad (2.13) \end{aligned}$$

the left Leibniz rule, for $\omega \in \Omega^1_\star$ and $f \in A_\star$. In particular:

$$\mathrm{d}_{{}^\star\nabla}(\omega_A \otimes_\star \mathrm{d}x^A) = (\mathrm{d}\omega_A - \omega_B \wedge_\star \Gamma^B_A) \otimes_\star \mathrm{d}x^A.$$

# NA differential geometry: NA torsion, NA curvature

Torsion:

$$\mathsf{T}^\star := \mathrm{d}_{\nabla^\star}\big(\partial_A \otimes_\star \mathrm{d}x^A\big) \ : \ \mathsf{Vec}_\star \otimes_\star \mathsf{Vec}_\star \to \mathsf{Vec}_\star,$$
$$\mathsf{T}^\star(\partial_A, \partial_B) = \partial_C \star (\Gamma^C_{AB} - \Gamma^C_{BA}) =: \partial_C \star \mathsf{T}^C{}_{AB}.$$

Torsion-free condition: $\Gamma^C_{AB} = \Gamma^C_{BA}$.

Curvature:

$$\mathsf{R}^\star := \mathrm{d}_{\nabla^\star} \bullet \mathrm{d}_{\nabla^\star} \ : \ \mathsf{Vec}_\star \ \longrightarrow \ \mathsf{Vec}_\star \otimes_\star \Omega^2_\star,$$
$$\mathsf{R}^\star(\partial_A) = \partial_C \otimes_\star (\mathrm{d}\Gamma^C_A + \Gamma^C_B \wedge_\star \Gamma^B_A) = \partial_C \otimes_\star \mathsf{R}^C_A,$$

with the curvature coefficients

$$\begin{aligned}
&\mathsf{R}^\star(\partial_A, \partial_B, \partial_C) = \langle\ \partial_D \otimes_\star \mathsf{R}^D_A\,,\ \partial_B \wedge_\star \partial_C\ \rangle_\star \\
&= \partial_D \star \big(\partial_C \Gamma^D_{AB} - \partial_B \Gamma^D_{AC} - \Gamma^D_{B'E} \star (\delta^E{}_B\, \Gamma^{B'}_{AC} + \mathrm{i}\,\kappa\, \mathscr{R}^{EG}{}_B\, (\partial_G \Gamma^{B'}_{AC})) \\
&+ \Gamma^D_{B'E} \star (\delta^E{}_C\, \Gamma^{B'}_{AB} + \mathrm{i}\,\kappa\, \mathscr{R}^{EG}{}_C\, (\partial_G \Gamma^{B'}_{AB}))\big) \\
&= \partial_D \star \mathsf{R}^D{}_{ABC}.
\end{aligned} \tag{2.14}$$

First Cartan structure equation:

$$\mathsf{T}^\star(u,v) = {}^{\phi_1}\nabla^\star_{\phi_2\,v}\left({}^{\phi_3}u\right) - {}^{\phi_1}\nabla^\star_{\phi_2\,{}_\alpha u}\left({}^{\phi_3\,\alpha}v\right) + [u,v]_\star. \tag{2.15}$$

Second Cartan structure equation:

$$\mathsf{R}^\star(z,u,v) = {}^{\kappa_1\,\check{\phi}_1\,\phi_1'}\boldsymbol{\nabla}^\star_{\bar{\rho}_3\,\bar{\zeta}_3\,\bar{\phi}_3\,\phi_3'\,v}\left({}^{\bar{\rho}_1\,\bar{\phi}_1\,\kappa_2\,\check{\phi}_2\,\phi_2'}\boldsymbol{\nabla}^\star_{\bar{\rho}_2\,\bar{\zeta}_2\,\check{\phi}_3\,u}{}^{\bar{\zeta}_1\,\bar{\phi}_2\,\kappa_3}z\right)$$
$$- {}^{\kappa_1\,\check{\phi}_1\,\phi_1'}\boldsymbol{\nabla}^\star_{\bar{\rho}_3\,\bar{\zeta}_3\,\bar{\phi}_3\,\phi_3'\,{}_\alpha u}\left({}^{\bar{\rho}_1\,\bar{\phi}_1\,\kappa_2\,\check{\phi}_2\,\phi_2'}\boldsymbol{\nabla}^\star_{\bar{\rho}_2\,\bar{\zeta}_2\,\check{\phi}_3\,\alpha\,v}{}^{\bar{\zeta}_1\,\bar{\phi}_2\,\kappa_3}z\right) + \boldsymbol{\nabla}^\star_{[u,v]_\star}z.$$

Bianchi identitites:

$$\begin{aligned}
&\mathrm{d}\mathsf{T}^A + \Gamma^A_B \wedge_\star \mathsf{T}^B = \mathsf{R}^A_B \wedge_\star \mathrm{d}x^B, \\
&\mathrm{d}\mathsf{R}^C_A + \Gamma^C_B \wedge_\star \mathsf{R}^B_A - \mathsf{R}^C_B \wedge_\star \Gamma^B_A = \\
&\qquad = \Gamma^C_B \wedge_\star \left(\Gamma^B_D \wedge_\star \Gamma^D_A\right) - {}^{\phi_1}\Gamma^C_B \wedge_\star \left({}^{\phi_2}\Gamma^B_D \wedge_\star {}^{\phi_3}\Gamma^D_A\right).
\end{aligned} \tag{2.16}$$

Ricci tensor:

$$\mathrm{Ric}^\star(u,v) := -\langle\ \mathrm{R}^\star(u,v,\partial_A)\,,\ \mathrm{d}x^A\ \rangle_\star \qquad (2.17)$$
$$\mathrm{Ric}^\star = \mathrm{Ric}_{AD} \star (\mathrm{d}x^D \otimes_\star \mathrm{d}x^A).$$

Commponents from $\mathrm{Ric}_{BC} := \mathrm{Ric}^\star(\partial_B, \partial_C)$

$$\begin{aligned} \mathrm{Ric}_{BC} &= \partial_A \Gamma^A_{BC} - \partial_C \Gamma^A_{BA} + \Gamma^A_{B'A} \star \Gamma^{B'}_{BC} - \Gamma^A_{B'C} \star \Gamma^{B'}_{BA} \\ &+ \mathrm{i}\,\kappa\, \Gamma^A_{B'E} \star \left(\mathscr{R}^{EG}{}_A\, (\partial_G \Gamma^{B'}_{BC}) - \mathscr{R}^{EG}{}_C\, (\partial_G \Gamma^{B'}_{BA})\right) \qquad (2.18) \\ &+ \mathrm{i}\,\kappa\, \mathscr{R}^{EG}{}_A\, \partial_G \partial_C \Gamma^A_{BE} - \mathrm{i}\,\kappa\, \mathscr{R}^{EG}{}_A\, \partial_G \left(\Gamma^A_{B'E} \star \Gamma^{B'}_{BC} - \Gamma^A_{B'C} \star \Gamma^{B'}_{BE}\right) \\ &+ \kappa^2\, \mathscr{R}^{AF}{}_D \left(\mathscr{R}^{EG}{}_A\, \partial_F (\Gamma^D_{B'E} \star \partial_G \Gamma^{B'}_{BC}) - \mathscr{R}^{EG}{}_C\, \partial_F (\Gamma^D_{B'E} \star \partial_G \Gamma^{B'}_{BA})\right)\ . \end{aligned}$$

Scalar curvature cannot be defined along these lines: cannot be seen as a map and inverse metric tensor needed. Not straightforward.

# NA deformation of GR: NA Levi-Civita connection

GR connection $\Gamma^{\mathsf{LC}\,\rho}_{\mu\nu}$ is a Levi-Civita connection: torssion-free and metric compatible $\nabla_\alpha g_{\mu\nu} = 0$. Generalization to our NA phase space:

Metric tensor $\mathrm{g}^\star \in \Omega^1_\star \otimes_\star \Omega^1_\star$ and $\mathrm{g}^\star(u,v) = \langle\, \mathrm{g}^\star\,,\, u \otimes_\star v\,\rangle_\star \in C^\infty(\mathcal{M})_\star$. In the coordinate basis:

$$\mathrm{g}^\star = \mathrm{g}_{AB} \star (\mathrm{d}x^A \otimes_\star \mathrm{d}x^B), \quad \mathrm{g}^\star(\partial_A, \partial_B) = \mathrm{g}_{AB} = \mathrm{g}_{BA}.$$

Metric compatibility condition: ${}^\star\nabla \mathrm{g}^\star = 0$. We calculate

$$\begin{aligned}
\mathrm{d}\mathrm{g}_{AB} &= \mathrm{d}\langle\, \mathrm{g}^\star\,,\, \partial_A \otimes_\star \partial_B\,\rangle_\star \\
&= \langle\, {}^\star\nabla \mathrm{g}^\star\,,\, \partial_A \otimes_\star \partial_B\,\rangle_\star + \langle\, {}^{\phi_1}\mathrm{g}^\star\,,\, {}^{\phi_2}\nabla^\star({}^{\phi_3}(\partial_A \otimes_\star \partial_B))\,\rangle_\star \\
&= \langle\, \mathrm{g}^\star\,,\, \nabla^\star(\partial_A \otimes_\star \partial_B)\,\rangle_\star .
\end{aligned}$$

Calcualtion straightforward up to

$$\mathsf{G}_{CN} \star \Gamma^N_{AD} = \tfrac{1}{2}\big(\partial_D \mathrm{g}_{AC} + \partial_A \mathrm{g}_{DC} - \partial_C \mathrm{g}_{AD} + \mathrm{i}\,\kappa\,\mathscr{R}^{EF}{}_C\,(\partial_E\partial_D \mathrm{g}_{AF} + \partial_E\partial_A \mathrm{g}_{DF})\big), \qquad (3.19)$$

$\mathsf{G}_{MN} = \mathrm{g}_{MN} + \mathrm{i}\,\kappa\,\mathscr{R}^{EF}{}_M\,\partial_E \mathrm{g}_{NF}$. Problems with straightforward inversion noted already in [Blumenhagen, Fuchs '16]. Working with $G^{MC} \star G_{CN} = \delta^M_N$ leads to:

$$G^{MC} \star (\mathsf{G}_{CN} \star \Gamma^N_{AD}) = ({}^{\bar\phi_1}G^{MC} \star {}^{\bar\phi_2}G_{CN}) \star {}^{\bar\phi_3}\Gamma^N_{AD} \neq \Gamma^M_{AD}. \qquad (3.20)$$

Inversion possible in terms of differential operators:

$$\Gamma^S_{AD} = \mathsf{G}^{*SC} * \mathsf{W}_{CAD} + \sum_{\vec\lambda} \frac{(\mathrm{i}\,\kappa)^{|\vec\lambda|}}{\vec\lambda\,!}\,(-1)^{I(\vec\lambda)}\,\mathrm{Y}_{\mathsf{G}}^{(\vec\lambda)}{}^S{}_M(\mathsf{G}^{*MC} * \mathsf{W}_{CAD}),$$

with $*$ being the $\star$-product corresponding to the twist

$$F = \exp\big(-\tfrac{\mathrm{i}\,\hbar}{2}(\partial_\mu \otimes \tilde\partial^\mu - \tilde\partial^\mu \otimes \partial_\mu)\big)$$

and $\mathsf{G}^{*SC} * \mathsf{G}_{CN} = \delta^S_N$.

Also:

$$\begin{aligned}
\mathrm{W}_{CAD} &= \frac{1}{2}\big(\partial_D \mathrm{g}_{AC} + \partial_A \mathrm{g}_{DC} - \partial_C \mathrm{g}_{AD} \\
&\quad + \mathrm{i}\,\kappa\,\mathscr{R}^{EF}{}_C\,(\partial_E\partial_D \mathrm{g}_{AF} + \partial_E\partial_A \mathrm{g}_{DF})\big),
\end{aligned}$$

$$\begin{aligned}
\mathrm{Y}_{\mathsf{G}}{}^M_N &= \overline{f}{}^{\beta}_R\big(\mathsf{G}^{*MC} * \overline{f}{}^{\alpha}_R(\mathsf{G}_{CN})\big)\,\overline{f}_{R\,\beta}\,\overline{f}_{R\,\alpha} =: \sum_{n=0}^{\infty} \frac{(\mathrm{i}\,\kappa)^n}{n!}\,\mathrm{Y}^{(n)}_{\mathsf{G}}{}^M_N, \\
\mathrm{Y}^{(0)}_{\mathsf{G}}{}^M_N &= \delta^M_N\ , \\
\mathrm{Y}^{(1)}_{\mathsf{G}}{}^M_N &= \frac{1}{2}\,\overline{f}{}^{\beta}_R\big(\mathsf{G}^{*MC} * (R^{\mu\nu\rho}\,p_\nu\,\partial_\rho \mathsf{G}_{CN})\big)\,\overline{f}_{R\,\beta}\,\partial_\mu \\
&\quad -\frac{1}{2}\,\overline{f}{}^{\beta}_R\big(\mathsf{G}^{*MC} * (\partial_\mu \mathsf{G}_{CN})\big)\,\overline{f}_{R\,\beta}\,R^{\mu\nu\rho}\,p_\nu\,\partial_\rho,
\end{aligned}$$

with $F_R^{-1} = \exp\big(\tfrac{\mathrm{i}\,\kappa}{2}\,R^{\mu\nu\rho}\,(p_\nu\,\partial_\rho \otimes \partial_\mu - \partial_\mu \otimes p_\nu\,\partial_\rho)\big) = \overline{f}{}^{\alpha}_R \otimes \overline{f}_{R\,\alpha}$.

Connection coefficients, expanded up to first order in $\hbar\kappa$:

$$
\begin{aligned}
\Gamma^{S(0,0)}_{AD} &= \Gamma^{\mathrm{LC}\,S}_{AD} = \tfrac{1}{2}\, g^{SQ} \left(\partial_D g_{AQ} + \partial_A g_{DQ} - \partial_Q g_{AD}\right)\,, \qquad (3.21)\\
\Gamma^{S(0,1)}_{AD} &= -\frac{\mathrm{i}\,\hbar}{2}\, g^{SP} \left((\partial_\mu g_{PQ})\,\tilde{\partial}^\mu \Gamma^{\mathrm{LC}\,Q}_{AD} - (\tilde{\partial}^\mu g_{PQ})\,\partial_\mu \Gamma^{\mathrm{LC}\,Q}_{AD}\right)\,,\\
\Gamma^{S(1,0)}_{AD} &= \mathrm{i}\,\kappa\, R^{\alpha\beta\gamma} \left(\tilde{g}^S{}_\gamma\, g_{\beta N}\left(\partial_\alpha \Gamma^{\mathrm{LC}\,N}_{AD}\right) - g^{SM}\, p_\beta\, (\partial_\gamma g_{MN})\,\partial_\alpha \Gamma^{\mathrm{LC}\,N}_{AD}\right)\,,\\
\Gamma^{S(1,1)}_{AD} &= \frac{\hbar\,\kappa}{2}\, R^{\alpha\beta\gamma} \Big[\ldots \text{ long expression } \ldots\\
&\qquad + (\partial_\alpha g^{SQ})\,(\partial_\beta g_{QP})\,\partial_\gamma \Gamma^{\mathrm{LC}\,P}_{AD}\Big].
\end{aligned}
$$

Comments:

-$\Gamma^{S(0,1)}_{AD}$ and $\Gamma^{S(1,0)}_{AD}$ imaginary, $\Gamma^{S(1,1)}_{AD}$ real.

-for $g_{MN}$ that does not depend on the momenta $p_\mu$, only the last term in $\Gamma^{S(1,1)}_{AD}$ remains.

-$\tilde{g}^S{}_\gamma = g^{SM}\,\delta_{M,\tilde{x}_\gamma}$.

# NA deformation of GR: NA vacuum Einstein equation

We can write vacuum Einstein equations in phase space as:

$$\mathrm{Ric}_{BC} = 0 \ . \tag{3.22}$$

Our strategy: expand Ricci tensor (2.17) in term of (3.21), i. e. the metric tensor $g_{MN}$. This gives Einstein equations in phase space. How do we obtain the induced equations in space-time?

From phase space to space-time:

- start from objects in space-time $M$ $g = g_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ and lift them to phase space $\mathcal{M}$ foliated with leaves of constant momenta, each leave is diffeomorphic to $M$.

$$
\begin{array}{ccc}
C^\infty(\mathcal{M}) & \xrightarrow{\;Q\;} & \widehat{C^\infty(\mathcal{M})} \\
\Big\uparrow{\scriptstyle \pi^*} & & \Big\downarrow{\scriptstyle s_{\bar{p}}^* = \sigma^*} \\
C^\infty(M) & \underset{Q_{\bar{p}}}{\dashrightarrow} & \widehat{C^\infty(M)}
\end{array}
$$

Functions, forms: pullback using the canonical projection $\pi : \mathcal{M} \to M$.

Vector fields: $v^\mu(x)\,\partial_\mu \mapsto \pi^*(v^\mu)(x,p)\,\partial_\mu$ and $\pi^*(v^\mu)(x,p) = (v^\mu)(\pi(x,p)) = v^\mu(x)$.

Metric tensor: $g = g_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \to \hat{\mathrm{g}}_{MN}\, \mathrm{d}x^M \otimes \mathrm{d}x^N$ with

$$(\hat{\mathrm{g}}_{MN}(x)) = \begin{pmatrix} \mathrm{g}_{\mu\nu}(x) & 0 \\ 0 & \mathrm{h}^{\mu\nu}(x) \end{pmatrix} . \tag{3.23}$$

Note the additional nondegenerate bilinear $\mathrm{h}(x)^{\mu\nu}\, \mathrm{d}\tilde{x}_\mu \otimes \mathrm{d}\tilde{x}_\nu$; natural choice $\mathrm{h}(x)^{\mu\nu} = \eta^{\mu\nu}$.

- Do all calculations in phase space, using the twisted differential geometry. In particular, calculate $\mathrm{Ric}_{BC}$ in terms of $g_{AB}$, (2.17), (3.21).
- Finally, project the result to space-time using the zero section $x \mapsto \sigma(x) = (x, 0)$.

  Functions, forms: pullback to the zero momentum leaf:

  Vector fields: $v^\mu(x,p)\, \partial_\mu + \tilde{v}_\mu(x,p)\, \tilde{\partial}^\mu \mapsto v^\mu(x,0)\, \partial_\mu$.

Ricci tensor: $\mathrm{Ric} \to \mathrm{Ric}^{\star\circ} = \mathrm{Ric}^\circ_{\mu\nu}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$,
$\mathrm{Ric}^\circ_{\mu\nu}(x) = \sigma^*(\mathrm{Ric}_{\mu\nu})(x,p) = \mathrm{Ric}_{\mu\nu}(x,0)$.

# NA deformation of GR: NA gravity in space-time

The lifted metric $\hat{\mathrm{g}}_{MN}\,\mathrm{d}x^M \otimes \mathrm{d}x^N = \mathrm{g}_{MN} \star (\mathrm{d}x^M \otimes_\star \mathrm{d}x^N)$,

$$\mathrm{g}_{MN}(x) = \begin{pmatrix} \mathrm{g}_{\mu\nu}(x) & \frac{\mathrm{i}\,\kappa}{2}\,R^{\sigma\nu\alpha}\,\partial_\sigma \mathrm{g}_{\mu\alpha} \\ \frac{\mathrm{i}\,\kappa}{2}\,R^{\sigma\mu\alpha}\,\partial_\sigma \mathrm{g}_{\alpha\nu} & \eta^{\mu\nu}(x) \end{pmatrix} . \tag{3.24}$$

Ricci tensor in space-time, (expanded up to first order in $\hbar\kappa$):

$$\begin{aligned} \mathrm{Ric}^\circ_{\mu\nu} = \mathrm{Ric}^{\mathrm{LC}}_{\mu\nu} + \tfrac{\ell_s^3}{12}\,R^{\alpha\beta\gamma}\Big(&\partial_\rho\big(\partial_\alpha \mathrm{g}^{\rho\sigma}\,(\partial_\beta \mathrm{g}_{\sigma\tau})\,\partial_\gamma \Gamma^{\mathrm{LC}\,\tau}_{\mu\nu}\big) \\ &-\partial_\nu\big(\partial_\alpha \mathrm{g}^{\rho\sigma}\,(\partial_\beta \mathrm{g}_{\sigma\tau})\,\partial_\gamma \Gamma^{\mathrm{LC}\,\tau}_{\mu\rho}\big) \\ +\,\partial_\gamma \mathrm{g}_{\tau\omega}\big(\partial_\alpha(\mathrm{g}^{\sigma\tau}\,&\Gamma^{\mathrm{LC}\,\rho}_{\sigma\nu})\,\partial_\beta \Gamma^{\mathrm{LC}\,\omega}_{\mu\rho} - \partial_\alpha(\mathrm{g}^{\sigma\tau}\,\Gamma^{\mathrm{LC}\,\rho}_{\sigma\rho})\,\partial_\beta \Gamma^{\mathrm{LC}\,\omega}_{\mu\nu} \\ +\,(\Gamma^{\mathrm{LC}\,\sigma}_{\mu\rho}\,\partial_\alpha \mathrm{g}^{\rho\tau} &- \partial_\alpha \Gamma^{\mathrm{LC}\,\sigma}_{\mu\rho}\,\mathrm{g}^{\rho\tau})\,\partial_\beta \Gamma^{\mathrm{LC}\,\omega}_{\sigma\nu} \\ -\,(\Gamma^{\mathrm{LC}\,\sigma}_{\mu\nu}\,\partial_\alpha \mathrm{g}^{\rho\tau} &- \partial_\alpha \Gamma^{\mathrm{LC}\,\sigma}_{\mu\nu}\,\mathrm{g}^{\rho\tau})\,\partial_\beta \Gamma^{\mathrm{LC}\,\omega}_{\sigma\rho}\big)\Big). \end{aligned} \tag{3.25}$$

Vacuum Einstein equations in space-time:

$$\mathrm{Ric}^\circ_{\mu\nu} = 0. \tag{3.26}$$

# NA deformation of GR: Comments

- $R$-flux (via NA differential geometry) generates non-trivial dynamical consequences on spacetime, they are independent of $\hbar$ and real-valued.
- the linear $R$-flux correction to $\mathrm{Ric}^{\circ}_{\mu\nu}$ is not a total derivative.
- Why zero momentum leaf? Pulling back to a leaf of constant momentum $p = p^{\circ}$ (generally) gives a non-vanishing imaginary contribution $\mathrm{Ric}^{(1,0)}_{\mu\nu}\big|_{p=p^{\circ}}$ to the spacetime Ricci tensor. Also, $n$-triproducts calculated on the zero momentum leaf [Aschieri, Szabo '15] coincide with those proposed in [Munich group '11].
- Why $\mathrm{h}(x)^{\mu\nu} = \eta^{\mu\nu}$? The simplest choice, can be extended. In relation with Born geometry [Freidel et al. '14]: in our model nonassociativity does not generates curved momentum space. Investigate $\mathrm{h}(x)^{\mu\nu} \neq \eta^{\mu\nu}\dots$

# Discussion

Our goals:

- Phenomenological consequences ($R$-flux induced corrections to GR solutions): to be investigates.
- Construction of scalar curvature, matter fields, full Einstein equations: to be investigated.
- Twisted diffeomorphism symmetry and "quantum diffeomorphisms": to be understood better.
- NA gravity as a gauge theory of Lorentz symmetry, NA Einstein-Cartan gravity: better understanding of NA gauge symmetry is needed, $L_\infty$ algebra? See also [Blumenhagen et al. '18].