# Axial gravity and split anomalies

Maro Cvitan
mcvitan@phy.hr

Faculty of Science, University of Zagreb

in collaboration with: L. Bonora, P. Dominis Prester, S. Giaccari, M. Paulišić and T. Štemberga

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# Introduction: metric-axial-tensor (MAT) gravity

usual coupling with gravity of the form:

$$h_{\mu\nu} T^{\mu\nu}$$

(here $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$)

we need different couplings for left and right matter:

$$h_{(L)\mu\nu} T_{(L)}{}^{\mu\nu} + h_{(R)\mu\nu} T_{(R)}{}^{\mu\nu}$$

the prescription:

$$\widehat{g}_{\mu\nu} = g_{\mu\nu} + \gamma_5 f_{\mu\nu}$$

we call this MAT or axial gravity

# Introduction: Bardeen's method I

Dirac fermions coupled to two Lie algebra valued gauge potentials (Bardeen 1969.)
vector $V_\mu$ and axial $A_\mu$

$$S[V,A] = i\int d^4x\,\overline{\psi}\left(\not{\partial} + \not{V} + \gamma_5 \not{A}\right)\psi$$

invariant under two sets of gauge transformations
two covariantly conserved currents $j_\mu = j^a_\mu T^a$ and $j_{5\mu} = j^a_{5\mu} T^a$, where

$$j^a_\mu = \overline{\psi}\gamma_\mu T^a \psi, \qquad j^a_{5\mu} = \overline{\psi}\gamma_\mu\gamma_5 T^a \psi$$

one-loop, e.g. vector conserved

$$[D^\mu_V j_\mu]^a + [A^\mu, j_{5\mu}]^a = 0$$

# Introduction: Bardeen's method II

the axial becomes anomalous:

$$
\begin{aligned}
[D_V^\mu j_{5\mu}]^a + [A^\mu, j_\mu]^a \quad = \quad & \frac{1}{4\pi^2}\varepsilon_{\mu\nu\lambda\rho}\mathrm{tr}\Bigg[T^a\Bigg(\frac{1}{4}F_V^{\mu\nu}F_V^{\lambda\rho} + \frac{1}{12}F_A^{\mu\nu}F_A^{\lambda\rho} \\
& -\frac{1}{6}F_V^{\mu\nu}A^\lambda A^\rho - \frac{1}{6}A^\mu A^\nu F_V^{\lambda\rho} \\
& -\frac{2}{3}A^\mu F_A^{\nu\lambda}A^\rho - \frac{1}{3}A^\mu A^\nu A^\lambda A^\rho\Bigg)\Bigg]
\end{aligned}
$$

taking the collapsing limit $V \to \frac{V}{2}, A \to \frac{V}{2}$ one infers that

$$
[D_{V\mu} j_L^\mu]^a = \frac{1}{24\pi^2}\varepsilon_{\mu\nu\lambda\rho}\mathrm{tr}\left[T^a\partial^\mu\left(V^\nu\partial^\lambda V^\rho + \frac{1}{2}V^\nu V^\lambda V^\rho\right)\right]
$$

where $j_{L\mu} = \overline{\psi}_L\gamma_\mu\psi_L$, here $\psi_L = \frac{1+\gamma_2}{2}\psi$, which is the consistent non-Abelian gauge anomaly

an example of split anomaly: opposite sign for $j_{R\mu}$

# Introduction: trace anomaly

at one loop the trace of the e.m. tensor receives corrections

$$\Theta_\mu{}^\mu = aE + cW^2 + eP$$

Weyl invariant (Weyl tensor squared):

$$W^2 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$$

Euler invariant (Gauss-Bonnet):

$$E = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

Pontryagin invariant:

$$P = \epsilon_{\mu\nu\rho\sigma}R^{\mu\nu\lambda\kappa}R^{\rho\sigma}{}_{\lambda\kappa}$$

in the sequel we use Bardeen's method in (axial) gravity, using collapsing limit to extract results for Weyl fermion:

$$\widehat{g}_{\mu\nu} = g_{\mu\nu} + \gamma_5 f_{\mu\nu} \qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$h_{\mu\nu} \to \frac{h_{\mu\nu}}{2} \qquad f_{\mu\nu} \to \frac{h_{\mu\nu}}{2}$$

# Contents

# Axial gravity: numbers

axial-complex numbers are defined by (similar to Hess, Greiner, 2009)

$$\hat{a} = a_1 + \gamma_5 a_2$$

a conjugation operator

$$\bar{\hat{a}} = a_1 - \gamma_5 a_2$$

axial-analytic functions $\hat{f}(\hat{x})$, where $\widehat{x} = x_1 + \gamma_5 x_2$
property:

$$\begin{aligned} \hat{f}(\hat{x}) &= P_+ \hat{f}(x_+) + P_- \hat{f}(x_-) \\ &= \frac{1}{2}\left(\hat{f}(x_+) + \hat{f}(x_-)\right) + \frac{\gamma_5}{2}\left(\hat{f}(x_+) - \hat{f}(x_-)\right) \end{aligned}$$

where $x_+ = x_1 + x_2$ and $x_- = x_1 - x_2$

# Axial gravity: numbers

derivatives of functions $f(\hat{x}, \bar{\hat{x}})$:

$$\frac{\partial}{\partial \hat{x}^\mu} = \frac{1}{2}\left(\frac{\partial}{\partial x_1^\mu} + \gamma_5 \frac{\partial}{\partial x_2^\mu}\right), \qquad \frac{\partial}{\partial \bar{\hat{x}}^\mu} = \frac{1}{2}\left(\frac{\partial}{\partial x_1^\mu} - \gamma_5 \frac{\partial}{\partial x_2^\mu}\right)$$

for axial-analytic functions $f(\hat{x})$ one has axial Cauchy-Riemann conditions $\frac{\partial f}{\partial \bar{\hat{x}}^\mu} = 0$ and therefore

$$\frac{d}{d\hat{x}} = \frac{\partial}{\partial x_1} \equiv \frac{\partial}{\partial \hat{x}},$$

as a consequence it follows that

$$\int d\hat{x}\, \widehat{f}(\hat{x}) = \int dx_1\, \widehat{f}(\hat{x})$$

and we can define definite integrals

$$\int_{\hat{a}}^{\hat{b}} d\hat{x}\, \widehat{f}(\hat{x}) = \widehat{g}(\hat{b}) - \widehat{g}(\hat{a})$$

# Axial gravity: geometry

metric $\widehat{g}_{\mu\nu} = g_{\mu\nu} + \gamma_5 f_{\mu\nu}$
the Christoffel symbols:

$$\begin{aligned}\widehat{\Gamma}^{\lambda}_{\mu\nu} &= \frac{1}{2}\widehat{g}^{\lambda\rho}\left(\frac{\partial}{\partial\widehat{x}^{\mu}}\widehat{g}_{\rho\nu} + \frac{\partial}{\partial\widehat{x}^{\nu}}\widehat{g}_{\mu\rho} - \frac{\partial}{\partial\widehat{x}^{\rho}}\widehat{g}_{\mu\nu}\right)\\ &= \Gamma^{(1)\lambda}_{\mu\nu} + \gamma_5\Gamma^{(2)\lambda}_{\mu\nu}\end{aligned}$$

using the axial affine parameter $\hat{t} = t_1 + \gamma_5 t_2$, the action for a particle is:

$$\widehat{S}[\widehat{x}] = \int d\hat{t}\left(\widehat{g}_{\mu\nu}\dot{\widehat{x}}^{\mu}\dot{\widehat{x}}^{\nu}\right)^{\frac{1}{2}}$$

where $\dot{\widehat{a}} = \frac{d\widehat{a}}{d\hat{t}}$.
the action takes values in axial complex numbers.
requiring $\delta\widehat{S}[\widehat{x}] = 0$ one gets the eom

$$\ddot{\widehat{x}}^{\mu} + \widehat{\Gamma}^{\mu}_{\nu\lambda}\dot{\widehat{x}}^{\nu}\dot{\widehat{x}}^{\lambda} = 0$$

# Point splitting: geodetic interval and distance

the quantity

$$\widehat{E} = \frac{1}{2}\widehat{g}_{\mu\nu}\dot{\widehat{x}}^{\mu}\dot{\widehat{x}}^{\nu}$$

is conserved as a function of $\hat{t}$. $\widehat{s} - \widehat{s}'$ is the axial arc length along the geodesic between $\widehat{x}$ and $\widehat{x}'$

$$\widehat{s} - \widehat{s}' = \int_{\hat{t}'}^{\hat{t}} d\hat{t}\,\sqrt{2\widehat{E}} = \sqrt{2\widehat{E}}\,(\hat{t} - \hat{t}').$$

the world function and denoted

$$\widehat{\sigma}(\widehat{x}, \widehat{x}') = \frac{1}{2}(\widehat{s} - \widehat{s}')^2$$

properties:

$$\widehat{\sigma}_{;\mu} = \widehat{\partial}_{\mu}\widehat{\sigma} = (\hat{t} - \hat{t}')\widehat{g}_{\mu\nu}\dot{\widehat{x}}^{\nu} \equiv -\widehat{g}_{\mu\nu}\widehat{y}^{\nu}$$

$$\frac{1}{2}\widehat{\sigma}_{;\mu}\widehat{\sigma}_{;}{}^{\mu} = \widehat{\sigma}$$

# Point splitting: normal coordinates

normal coordinates can be defined based at $x$ or at $x'$:

$$\widehat{y}^{\mu'}(\widehat{x}',\widehat{x}) = (\hat{t}-\hat{t}')\frac{d\widehat{x}^{\mu'}}{d\hat{t}'} \quad \text{and} \quad \widehat{y}^{\mu}(\widehat{x},\widehat{x}') = (\hat{t}'-\hat{t})\frac{d\widehat{x}^{\mu}}{d\hat{t}}$$

using geodesic equation at endpoints it follows that

$$\widehat{y}^{\mu'}{}_{;\nu}(\hat{x}',\hat{x})\widehat{y}^{\nu}(\hat{x},\hat{x}') \quad = \quad -\widehat{y}^{\mu'}(\widehat{x}',\widehat{x})$$

also it folllows that

$$\widehat{y}^{\mu'}{}_{;\nu}(\widehat{x}',\widehat{x})\frac{d\widehat{x}^{\nu}(\hat{t})}{d\hat{t}} = \frac{d\widehat{x}^{\mu'}(\hat{t}')}{d\hat{t}'}$$

in the coincidence limit $\widehat{x}' \to \widehat{x}$ one gets

$$[\widehat{y}^{\mu}] = 0 \qquad [\widehat{y}^{\mu'}] = 0$$

$$[\widehat{y}^{\mu'}{}_{;\nu}] = \delta^{\mu}_{\nu} \qquad [\widehat{y}^{\mu'}{}_{;\nu'}] = -\delta^{\mu}_{\nu} \qquad [\widehat{y}^{\mu}{}_{;\nu}] = -\delta^{\mu}_{\nu} \qquad [\widehat{y}^{\mu}{}_{;\nu'}] = \delta^{\mu}_{\nu}$$

where $[X]$ denotes the result of the coincidence limit on the quantity $X$

# Point splitting: normal coordinates

differentiating several times with respect to $\widehat{x}^{\rho}$ or $\widehat{x}^{\rho'}$ one finds further relations for coincidence limits

$$[\widehat{y}^{\mu'}{}_{;\lambda'\rho'}] = 0$$

$$[\widehat{y}^{\mu'}{}_{;\lambda'\rho'\tau'}] = \frac{1}{3}\left(\widehat{R}^{\mu}{}_{\rho\lambda\tau} + \widehat{R}^{\mu}{}_{\tau\lambda\rho}\right)$$

$$[\widehat{y}^{\mu'}{}_{;\lambda\rho\tau}] = \frac{1}{3}\left(\widehat{R}^{\mu}{}_{\lambda\rho\tau} + \widehat{R}^{\mu}{}_{\rho\lambda\tau}\right)$$

$$[\widehat{y}^{\mu}{}_{;\lambda\rho\tau}] = \frac{1}{3}\left(\widehat{R}^{\mu}{}_{\tau\lambda\rho} + \widehat{R}^{\mu}{}_{\rho\lambda\tau}\right)$$

# Point splitting: coincidence limits of $\widehat{\sigma}$

covariantly differentiating properties of $\sigma$

$$\widehat{\sigma}_{;\nu} = \widehat{\sigma}_{;\mu\nu}\widehat{\sigma}_{;}{}^{\mu} \qquad \widehat{\sigma}_{;\mu\lambda} = -\widehat{g}_{\mu\nu}\,\widehat{y}^{\nu}{}_{;\lambda}$$

it follows

$$[\widehat{\sigma}_{;\nu}] = 0 \qquad [\widehat{\sigma}_{;\mu\lambda}] = \widehat{g}_{\mu\lambda} \qquad [\widehat{\sigma}_{;\mu\lambda'}] = -\widehat{g}_{\mu\lambda}$$

$$[\widehat{\sigma}_{;\rho\nu\lambda}] = [\widehat{\sigma}_{;\lambda\nu\rho}] = [\widehat{\sigma}_{;\nu\lambda\rho}] = 0$$

$$[\widehat{\sigma}_{;\nu\lambda\rho\tau}] = -\frac{1}{3}\left(\widehat{R}_{\nu\tau\lambda\rho} + \widehat{R}_{\nu\rho\lambda\tau}\right) \equiv \widehat{S}_{\nu\lambda\rho\tau}$$

$$[\widehat{\sigma}_{;\nu\lambda\rho\sigma\tau}] = \frac{3}{4}\left(\widehat{S}_{\nu\lambda\sigma\tau;\rho} + \widehat{S}_{\nu\lambda\sigma\rho;\tau} + \widehat{S}_{\nu\lambda\tau\rho;\sigma}\right)$$

# Point splitting: coincidence limits of $\widehat{\sigma}$

$$[\widehat{\sigma}_{;\mu\nu'\lambda}] = [\widehat{\sigma}_{;\mu\lambda\nu'}] = 0$$

$$[\widehat{\sigma}_{;\mu\nu'\lambda\rho}] \quad = \quad [\widehat{\sigma}_{;\mu\lambda\rho\nu'}] = -[\widehat{\sigma}_{;\mu\lambda\rho\nu}] = -\widehat{S}_{\mu\lambda\rho\nu}$$

and

$$[\widehat{\sigma}_{;\mu\nu'\lambda\rho\sigma}] = [\widehat{\sigma}_{;\mu\lambda\rho\sigma\nu'}] = \frac{1}{4}\widehat{S}_{\mu\lambda\rho\sigma;\nu} - \frac{3}{4}\left(\widehat{S}_{\mu\lambda\nu\rho;\sigma} + \widehat{S}_{\mu\lambda\sigma\nu;\rho}\right)$$

similarly, one obtains

$$[\widehat{\sigma}_{;\mu}{}^{\mu}{}_{\nu}{}^{\nu}{}_{\rho}{}^{\rho}] = -\frac{8}{5}\widehat{R}_{;\mu}{}^{\mu} + \frac{4}{15}\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu} - \frac{4}{15}\widehat{R}_{\mu\nu\lambda\rho}\widehat{R}^{\mu\nu\lambda\rho}$$

$$[\widehat{\sigma}_{;\mu\nu}{}^{\nu}{}_{\rho}{}^{\rho\mu}] = -[\widehat{\sigma}_{;\mu}{}^{\mu'}{}_{\nu}{}^{\nu}{}_{\rho}{}^{\rho}] = \frac{2}{5}\widehat{R}_{;\mu}{}^{\mu} - \frac{1}{15}\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu} - \frac{4}{15}\widehat{R}_{\mu\nu\lambda\rho}\widehat{R}^{\mu\nu\lambda\rho}$$

the Van Vleck-Morette determinant in MAT is defined by

$$\widehat{D}(\widehat{x},\widehat{x}') = \det(-\widehat{\sigma}_{;\mu\nu'})$$

weight 0 version is $\widehat{\Delta}(\widehat{x},\widehat{x}') = \frac{1}{\sqrt{\widehat{g}(\widehat{x})}}\widehat{D}(\widehat{x},\widehat{x}')\frac{1}{\sqrt{\widehat{g}(\widehat{x}')}}$

in the coincidence limit: $[\widehat{\Delta}^{\frac{1}{2}}_{;\lambda}] = 0$

$$[\widehat{\Delta}^{\frac{1}{2}}_{;\lambda\rho}] = \frac{1}{6}\widehat{R}_{\lambda\rho} = \frac{1}{6}\left(R^{(1)}_{\lambda\rho} + \gamma_5 R^{(2)}_{\lambda\rho}\right)$$

$$[\widehat{\Delta}^{\frac{1}{2}}_{;\lambda\rho\sigma}] = \frac{1}{12}\left(\widehat{R}_{\lambda\rho;\sigma} + \widehat{R}_{\rho\sigma;\lambda} + \widehat{R}_{\sigma\lambda;\rho}\right)$$

$$[\widehat{\Delta}^{\frac{1}{2}}{}_{;\mu}{}^{\mu}{}_{\nu}{}^{\nu}] = +\frac{1}{5}\widehat{R}_{;\mu}{}^{\mu} + \frac{1}{36}\widehat{R}^2 - \frac{1}{30}\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu} + \frac{1}{30}\widehat{R}_{\mu\nu\lambda\rho}\widehat{R}^{\mu\nu\lambda\rho}$$

# Point splitting: parallel displacement matrix

$I(x,x')\psi(x')$ is the spinor at $x$ obtained by parallel displacement of $\psi(x')$ along the geodesic from $x'$ to $x$. $I(x,x')$ is a bispinor quantity satisfying

$$\widehat{\sigma}_{;}{}^{\mu}\widehat{I}_{;\mu} = 0, \qquad [\widehat{I}] = \mathbf{1}$$

and $\mathbf{1}$ is the identity matrix in the spinor space.
further properties:

$$[\widehat{I}_{;(\mu\nu)}] = 0$$

$$[\widehat{I}_{;[\mu\nu]}] = -\frac{1}{4}\widehat{\mathcal{R}}_{\mu\nu} \qquad \text{where } \widehat{\mathcal{R}}_{\mu\nu} = \widehat{R}_{\mu\nu}{}^{ab}\Sigma_{ab}$$

$$[\widehat{I}_{;\nu}{}^{\nu}{}_{\rho}] = \frac{1}{6}\widehat{\nabla}^{\nu}\widehat{\mathcal{R}}_{\rho\nu} \qquad [\widehat{I}_{;\nu}{}^{\nu}{}_{\rho}{}^{\rho}] = \frac{1}{8}\widehat{\mathcal{R}}_{\rho\lambda}\widehat{\mathcal{R}}^{\rho\lambda}$$

# Fermions in axial background: the action

fermion interacting with a metric and an axial tensor:

$$\begin{aligned}
\widehat{S} &= \int d^4\widehat{x}\,\left(i\overline{\psi}\sqrt{\overline{\widehat{g}}}\gamma^a\widehat{e}_a^\mu\left(\partial_\mu+\frac{1}{2}\widehat{\Omega}_\mu\right)\psi\right)(\widehat{x}) \\
&= \int d^4\widehat{x}\,\left(i\overline{\psi}\sqrt{\overline{\widehat{g}}}\gamma^a(\tilde{e}_a^\mu+\gamma_5\tilde{c}_a^\mu)\left(\partial_\mu+\frac{1}{2}\left(\Omega_\mu^{(1)}+\gamma_5\Omega_\mu^{(2)}\right)\right)\psi\right)(\widehat{x}) \\
&= \int d^4\widehat{x}\,\left(i\overline{\psi}\sqrt{\overline{\widehat{g}}}(\tilde{e}_a^\mu-\gamma_5\tilde{c}_a^\mu)\left[\frac{1}{2}\gamma^a\overleftrightarrow{\partial}_\mu+\frac{1}{4}\left(\gamma^a\widehat{\Omega}_\mu+\overline{\widehat{\Omega}}_\mu\gamma^a\right)\right]\psi\right)(\widehat{x}) \\
&= \int d^4\widehat{x}\,\left(i\overline{\psi}\sqrt{\overline{\widehat{g}}}(\tilde{e}_a^\mu-\gamma_5\tilde{c}_a^\mu)\left[\frac{1}{2}\gamma^a\overleftrightarrow{\partial}_\mu+\frac{i}{4}\gamma_d\epsilon^{dabc}\widehat{\Omega}_{\mu bc}\gamma_5\right]\psi\right)(\widehat{x})
\end{aligned}$$

takes axial-real values

notation: here $\partial_\mu = \frac{\partial}{\partial\widehat{x}^\mu}$ applies only to $\psi$ or $\overline{\psi}$

the action is invariant under axially extended diffeomorphisms

# Fermions in axial background: Ward identities

action Weyl invariant with axial Weyl transformation for the fermion field:

$$\psi \to e^{-\frac{3}{2}(\omega+\gamma_5\eta)}\psi$$

action of the form $\overline{\psi}\mathcal{O}\psi$
the Ward identity

$$0 = \int \overline{\psi}\frac{\delta\mathcal{O}}{\delta\widehat{g}_{\mu\nu}}\widehat{g}_{\mu\nu}\,(\omega+\gamma_5\eta)\psi$$

one gets in this way two WI's

$$\begin{aligned} \mathcal{T}(x) &\equiv T^{\mu\nu}g_{\mu\nu} + T_5^{\mu\nu}f_{\mu\nu} = 0, \\ \mathcal{T}_5(x) &\equiv T^{\mu\nu}f_{\mu\nu} + T_5^{\mu\nu}g_{\mu\nu} = 0, \end{aligned}$$

# Fermions in axial background: EM tensor

defined as:

$$\mathbf{T}^{\mu\nu} = \frac{2}{\sqrt{\widehat{g}}} \frac{\delta S}{\delta \widehat{g}_{\mu\nu}}$$

here the operator $\frac{2}{\sqrt{\widehat{g}}} \frac{\delta}{\delta \widehat{g}_{\mu\nu}}$ acts from the right and inside $\overline{\psi} \dots \psi$.

$$\mathbf{T}^{\lambda\rho} = -\frac{i}{2} \overline{\psi} \widehat{\gamma}^{\lambda} \widehat{\nabla}^{\rho} \psi + (\lambda \leftrightarrow \rho)$$

# Schwinger-DeWitt proper time method: fermion propagator

one starts from the fermion propagator

$$\widehat{G}(\widehat{x},\widehat{x}') = \langle 0|\mathcal{T}\psi(\widehat{x})\psi^\dagger(\widehat{x}')|0\rangle$$

which satisfies

$$i\sqrt{\widehat{g}}\gamma_0\,\widehat{\gamma}^\mu\widehat{\nabla}_\mu\widehat{G}(\widehat{x},\widehat{x}') = -\mathbf{1}\delta(\widehat{x},\widehat{x}')$$

using the ansatz

$$\widehat{G}(x,x') = -i\overline{\widehat{\gamma}}^\mu\overline{\widehat{\nabla}}_\mu\overline{\widehat{\mathcal{G}}}(x,x')\gamma_0^{-1}$$

gives

$$\sqrt{|\widehat{g}|}\underbrace{\left(\overline{\widehat{\nabla}}_\mu\overline{\widehat{g}}^{\mu\nu}\overline{\widehat{\nabla}}_\nu - \frac{1}{4}\overline{\widehat{R}}\right)}_{\overline{\widehat{\mathcal{F}}}_{\hat{x}}}\overline{\widehat{\mathcal{G}}}(\widehat{x},\widehat{x}') = -\mathbf{1}\delta(\widehat{x},\widehat{x}')$$

we consider the amplitude:

$$K(\widehat{x},\widehat{x}',\widehat{s}) \equiv \langle \widehat{x},\widehat{s}|\widehat{x}',0\rangle = \langle \widehat{x}|e^{i\widehat{\mathfrak{F}}\widehat{s}}|\widehat{x}'\rangle$$

which satisfies the (heat kernel) differential equation

$$i\frac{\partial}{\partial\widehat{s}}\langle \widehat{x},\widehat{s}|\widehat{x}',0\rangle = -\widehat{\mathfrak{F}}_{\widehat{x}}\langle \widehat{x},\widehat{s}|\widehat{x}',0\rangle$$

where $\widehat{\mathfrak{F}}_{\widehat{x}}$ is the differential operator

$$\widehat{\mathfrak{F}}_{\widehat{x}} = \widehat{\nabla}_\mu \widehat{g}^{\mu\nu}\widehat{\nabla}_\nu - \frac{1}{4}\widehat{R}$$

then we make the ansatz

$$\langle \widehat{x},\widehat{s}|\widehat{x}',0\rangle = -\lim_{m\to 0}\frac{i}{16\pi^2}\frac{\sqrt{\widehat{D}(\widehat{x},\widehat{x}')}}{\widehat{s}^2}e^{i\left(\frac{\widehat{\sigma}(\widehat{x},\widehat{x}')}{2\widehat{s}}-m^2\widehat{s}\right)}\widehat{\Phi}(\widehat{x},\widehat{x}',\widehat{s})$$

$\widehat{\Phi}(\widehat{x},\widehat{x}',\widehat{s})$ is a function to be determined
$m$ will eventually be set to zero

# Schwinger-DeWitt proper time method: heat kernel II

equation for $\widehat{\Phi}(\widehat{x},\widehat{x}',\widehat{s})$ is

$$i\frac{\partial\widehat{\Phi}}{\partial\widehat{s}}+\frac{i}{\widehat{s}}\widehat{\nabla}^{\mu}\widehat{\Phi}\widehat{\nabla}_{\mu}\widehat{\sigma}+\frac{1}{\sqrt{\widehat{D}}}\widehat{\nabla}^{\mu}\widehat{\nabla}_{\mu}\left(\sqrt{\widehat{D}}\widehat{\Phi}\right)-\left(\frac{1}{4}\widehat{R}-m^2\right)\widehat{\Phi}=0$$

with boundary condition at $\widehat{s}\to 0$

$$\lim_{\widehat{s}\to 0}\widehat{\Phi}(\widehat{x},\widehat{x}',\widehat{s})=\mathbf{1}$$

expanding

$$\widehat{\Phi}(\widehat{x},\widehat{x}',\widehat{s})=\sum_{n=0}^{\infty}\widehat{a}_n(\widehat{x},\widehat{x}')(i\widehat{s})^n$$

equations for $\widehat{a}_n$ become:

$$(n+1)\widehat{a}_{n+1}+\widehat{\nabla}^{\mu}\widehat{a}_{n+1}\widehat{\nabla}_{\mu}\widehat{\sigma}-\frac{1}{\sqrt{\widehat{D}}}\widehat{\nabla}^{\mu}\widehat{\nabla}_{\mu}\left(\sqrt{\widehat{D}}\widehat{a}_n\right)+\left(\frac{1}{4}\widehat{R}-m^2\right)\widehat{a}_n=0$$

with boundary condition $[\widehat{a}_0]=\mathbf{1}$

# Schwinger-DeWitt proper time method: heat kernel III

$$(n+1)\widehat{a}_{n+1} + \widehat{\nabla}^{\mu}\widehat{a}_{n+1}\widehat{\nabla}_{\mu}\widehat{\sigma} - \frac{1}{\sqrt{\widehat{D}}}\widehat{\nabla}^{\mu}\widehat{\nabla}_{\mu}\left(\sqrt{\widehat{D}}\widehat{a}_n\right) + \left(\frac{1}{4}\widehat{R} - m^2\right)\widehat{a}_n = 0$$

one finds

$$[\widehat{a}_1] = \left(-\frac{1}{12}\widehat{R} + m^2\right)\mathbf{1}$$

$$\begin{aligned}[\widehat{a}_2] \quad = \quad & \left(\frac{1}{2}m^4 - \frac{1}{12}m^2\widehat{R} + \frac{1}{288}\widehat{R}^2 - \frac{1}{120}\widehat{R}_{;\mu}{}^{\mu}\right. \\ & \left. -\frac{1}{180}\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu} + \frac{1}{180}\widehat{R}_{\mu\nu\lambda\rho}\widehat{R}^{\mu\nu\lambda\rho} + \frac{1}{48}\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\right)\mathbf{1}\end{aligned}$$

# Schwinger-DeWitt proper time method: effective action I

to proceed we identify the effective action for Dirac fermions with

$$\widehat{W} = -\frac{i}{2}\mathrm{Tr}\left(\ln\,\widehat{\mathcal{F}}\right)$$

its axially extended Weyl variation is given by

$$\delta_{\widehat{\omega}}\widehat{W} = \frac{i}{2}\mathrm{Tr}\left(\widehat{\mathcal{G}}\,\delta_{\widehat{\omega}}\widehat{\mathcal{F}}\right)$$

we have

$$\delta_{\widehat{\omega}}\widehat{W} = \delta_{\widehat{\omega}}\left(-\frac{1}{2}\mathrm{Tr}\int_0^\infty \frac{d\widehat{s}}{i\widehat{s}}e^{i\widehat{\mathcal{F}}\,\widehat{s}}\right) = -\frac{1}{2}\mathrm{Tr}\left(\int_0^\infty d\widehat{s}\,e^{i\widehat{\mathcal{F}}\,\widehat{s}}\delta_{\widehat{\omega}}\widehat{\mathcal{F}}\right).$$

for our purposes, the effective action can be represented as

$$\widehat{W} = -\frac{1}{2}\mathrm{Tr}\int_0^\infty \frac{d\widehat{s}}{i\widehat{s}}e^{i\widehat{\mathcal{F}}\widehat{s}} + \mathrm{const} \equiv \widehat{L} + \mathrm{const}$$

$$\widehat{W} = -\frac{1}{2}\mathrm{Tr}\int_0^\infty \frac{d\widehat{s}}{i\widehat{s}} e^{i\widehat{\mathcal{F}}\widehat{s}} + \mathrm{const} \equiv \widehat{L} + \mathrm{const}$$

here $\widehat{L}$ is the relevant effective action

$$\widehat{L} = \int d^d\widehat{x}\,\widehat{L}(\widehat{x})$$

where

$$\widehat{L}(\widehat{x}) = -\frac{1}{2}\mathrm{tr}\int_0^\infty \frac{d\widehat{s}}{i\widehat{s}}\widehat{K}(\widehat{x},\widehat{x},\widehat{s})$$

$$\widehat{K}(\widehat{x},\widehat{x},\widehat{s}) = \frac{i}{(4\pi i\widehat{s})^{\frac{d}{2}}}\sqrt{\widehat{g}}\, e^{-im^2\widehat{s}}[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})]$$

# Schwinger-DeWitt proper time method: dimensional regularisation I

to analytically continue in number of dimensions $d$ we multiply $\widehat{L}$ with a appropriate power of mass parameter $\mu$ to make it dimensionless

$$\frac{\widehat{L}(x)}{\mu^d} = -\frac{i}{2}(4\pi\mu^2)\mathrm{tr}\int_0^\infty d\widehat{s}\,(4\pi i\mu^2\widehat{s})^{-\frac{d}{2}-1}\sqrt{\widehat{g}}\,e^{-im^2\widehat{s}}[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})]$$

where $\mathrm{tr}$ denotes the trace over gamma matrices.
we assume that

$$\lim_{s\to\infty} e^{-im^2\widehat{s}}[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})] = 0$$

integrate by parts

$$\frac{\widehat{L}(x)}{\mu^d} = -\frac{4i}{d(2-d)(4-d)}\frac{1}{(4\pi\mu^2)^2}\mathrm{tr}\int_0^\infty d\widehat{s}\,(4\pi i\mu^2\widehat{s})^{2-\frac{d}{2}}\sqrt{\widehat{g}}\;\frac{\partial^3}{\partial(i\widehat{s})^3}\left(e^{-im^2\widehat{s}}[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})]\right)$$

# Schwinger-DeWitt proper time method: dimensional regularisation II

now we can plug in:

$$[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})] = 1 + [\widehat{a}_1] i\widehat{s} + [\widehat{a}_2](i\widehat{s})^2 + \dots$$

expand around $d = 4$:

$$\begin{aligned} \widehat{L}(\widehat{x}) \quad \approx \quad & \frac{1}{32\pi^2}\left(\frac{1}{d-4} - \frac{3}{4}\right) \mathrm{tr}\left(\left(m^4 - 2m^2[\widehat{a}_1] + 2[\widehat{a}_2]\right)\sqrt{\widehat{g}}\right) \\ & + \frac{i}{64\pi^2}\mathrm{tr}\int_0^\infty d\widehat{s}\, \ln(4\pi i\mu^2\widehat{s})\sqrt{\widehat{g}}\frac{\partial^3}{\partial(i\widehat{s})^3}\left(e^{-im^2\widehat{s}}[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})]\right) \end{aligned}$$

the last line does not contain singularities in $d \to 4$. it depends explicitly on the parameter $\mu$ and represent a nonlocal part. its variation with respect to $\widehat{\omega}$ gives the trace of renormalised e.m. tensor. the first line contains the anomaly.

# Schwinger-DeWitt proper time method: the anomaly I

$$[\widehat{a}_1] = \left(-\frac{1}{12}\widehat{R} + m^2\right)\mathbf{1}$$

$$\begin{aligned}[\widehat{a}_2] \quad = \quad & \left(\frac{1}{2}m^4 - \frac{1}{12}m^2\widehat{R} + \frac{1}{288}\widehat{R}^2 - \frac{1}{120}\widehat{R}_{;\mu}{}^{\mu}\right. \\ & \left. -\frac{1}{180}\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu} + \frac{1}{180}\widehat{R}_{\mu\nu\lambda\rho}\widehat{R}^{\mu\nu\lambda\rho} + \frac{1}{48}\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\right)\mathbf{1}\end{aligned}$$

one puts $m = 0$
we are only interested in the odd parity part of the anomaly which can only come from the term $\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}$ contained in $[a_2]$
the $\frac{1}{d-4}$ singularity is canceled by terms in the variation of the first line
the only relevant contribution is:

$$\frac{1}{768\pi^2}\mathrm{tr}\int d^4x\,\sqrt{\widehat{g}}\,\widehat{\omega}\,\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}$$

we write the variation of the second line in terms of renormalised energy momentum tensor $\widehat{\Theta}_{\mu\nu}$

$$\mathrm{tr} \int d^4x \sqrt{\widehat{g}}\, \widehat{\omega}\, \widehat{\Theta}_{\mu}{}^{\mu}$$

comparing and taking into account $\widehat{\Theta}_{\mu}{}^{\mu} = \Theta_{\mu}{}^{\mu} + \gamma_5 \Theta_{5\mu}{}^{\mu}$

$$\begin{aligned}
\Theta_{\mu}{}^{\mu}(x) &= -\frac{1}{768\pi^2} \mathrm{tr}\left(\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\right)\Big|_{\mathrm{odd}} = -\frac{2i}{768\pi^2}\epsilon^{\mu\nu\lambda\rho} R^{(1)}_{\mu\nu\alpha\beta} R^{(2)\alpha\beta}_{\lambda\rho} \\
\Theta_{5\mu}{}^{\mu}(x) &= -\frac{1}{768\pi^2} \mathrm{tr}\left(\gamma_5\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\right)\Big|_{\mathrm{odd}} \\
&= -\frac{i}{768\pi^2}\epsilon^{\mu\nu\lambda\rho}\left(R^{(1)}_{\mu\nu\alpha\beta} R^{(1)\alpha\beta}_{\lambda\rho} + R^{(2)}_{\mu\nu\alpha\beta} R^{(2)\alpha\beta}_{\lambda\rho}\right)
\end{aligned}$$

# Schwinger-DeWitt proper time method: the collapsing limit

the collapsing limit $h_{\mu\nu} \to \frac{h_{\mu\nu}}{2}, f_{\mu\nu} \to \frac{h_{\mu\nu}}{2}$

$$\widehat{e}^a_m \to \delta^a_m \frac{1-\gamma_5}{2} + e^a_m \frac{1+\gamma_5}{2}, \qquad \widehat{e}^m_a \to \delta^m_a \frac{1-\gamma_5}{2} + e^m_a \frac{1+\gamma_5}{2}$$

$$\sqrt{\widehat{g}} \to \frac{1-\gamma_5}{2}\sqrt{\eta} + \frac{1+\gamma_5}{2}\sqrt{g}$$

$$\Gamma^{(1)\lambda}_{\mu\nu} \to \frac{1}{2}\Gamma^{\lambda}_{\mu\nu}, \qquad \Gamma^{(2)\lambda}_{\mu\nu} \to \frac{1}{2}\Gamma^{\lambda}_{\mu\nu}$$

$$\Omega^{(1)ab}_{\mu} \to \frac{1}{2}\omega^{ab}_{\mu}, \qquad \Omega^{(2)ab}_{\mu} \to \frac{1}{2}\omega^{ab}_{\mu}$$

$$R^{(1)}_{\mu\nu\lambda}{}^{\rho} \to \frac{1}{2}R_{\mu\nu\lambda}{}^{\rho}, \qquad R^{(2)}_{\mu\nu\lambda}{}^{\rho} \to \frac{1}{2}R_{\mu\nu\lambda}{}^{\rho}$$

$$S = \int d^4x \left[ i\overline{\psi}\gamma^{\mu}\frac{1-\gamma_5}{2}\partial_{\mu}\psi + \int d^4x\,\sqrt{g}\,i\overline{\psi}\gamma^a e^{\mu}_a \left(\partial_{\mu} + \frac{1}{2}\omega_{\mu}\right)\frac{1+\gamma_5}{2}\psi \right]$$

$$\Theta_{\mu}{}^{\mu}(x) = -\frac{i}{1536\pi^2}\epsilon^{\mu\nu\lambda\rho}R_{\mu\nu\alpha\beta}R_{\lambda\rho}{}^{\alpha\beta}$$

# Conclusion

- calculations of split anomalies proven to be difficult. need to deal with Weyl fermions. functional integration measure definition for Weyl fermions problematic.
- disagreement in the literature (Bastianelli, Martelli 2016 find no anomaly).
- the approach presented here uses path integral measure for Dirac fermions that is well defined, uses axial gravity to keep chiralities separate. square of the Dirac operator used here preserves axially extended diffeomorphisms.
- results presented here support the existence of the anomaly.
- existence of the anomaly forbids the fundamantal theories containing Weyl fermions with inequal number of left and right chiralities.

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