Differential calculus on Jordan algebras and Jordan modules

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Observables of any quantum system are given by hermitian (self-adjoint) elements in a **C* algebra** ($\subseteq \mathcal{B}(\mathcal{H})$ for a separable Hilbert space $\mathcal{H}$). Nevertheless

"The basic operations on matrices or operators are multiplication by a complex scalar, addition, multiplication of matrices (composition of operators), and forming the complex conjugate transpose matrix (adjoint operator).
This formalism is open to the objection that the operations are not "observable," not intrinsic to the physically meaningful part of the system.
In 1932 *Jordan proposed a program to discover a new algebraic setting for quantum mechanics, which would be freed from dependence on an invisible but all-determining metaphysical matrix structure."*

K. Mc Crimmon, *A Taste of Jordan Algebras*

Jordan algebras were introduced in the 30's to formalize properties of a finite dimensional quantum system.

We must understand what are the essential operations to perform on physical observables:

- The multiplication by a real scalar
- The sum of two observables
- Powers of observables
- Identity

Also we require formal reality:

$$x^2 + y^2 = 0 \Rightarrow x = y = 0.$$

Define the product $x \circ y = \frac{1}{2}\left((x+y)^2 - x^2 - y^2\right)$.

Theorem (Jordan 1932)

Let J be a real vector such that

$$x \in J \to \exists\, x^n \in J \;\forall\, n \in \mathbb{N}.$$

then the following are equivalent:

1. $x^r \circ x^s = x^{r+s}$ *(Power associativity)*
2. $x \circ (y \circ x^2) = (x \circ y) \circ x^2$. *(**Jordan Identity**)*

Definition

A **Jordan algebra** $(J, \circ)$, *is a vector space* J *together with a bilinear product* $\circ : J \times J \to J$, *such that* $\forall x, y \in J$:

- $x \circ y = y \circ x$
- $(x \circ y) \circ x^2 = x \circ (y \circ x^2)$

We will always consider unital Jordan algebras (Which is always the case for Euclidean Jordan algebras).

Example (Special Jordan algebras)

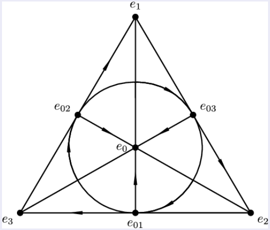
Let A be an associative algebra, equip it with the product:

$$x \circ y = \frac{1}{2}(xy + yx).$$

$(A, \circ)$ *is a Jordan algebra. Every Jordan algebra isomorphic to an algebra of this kind is called a* **special Jordan algebra***.*
Notice that if A is a $*-$algebra and A_{sa} is the subspace of self-adjoint elements of A, then A_{sa} is not a subalgebra of A but it is a Jordan subalgebra of $(A, \circ)$.

Example (The exceptional Jordan algebra)

The multiplication diagram of octonions:



The **exceptional Jordan algebra** *(***Albert algebra***)*

$$J_3^8 = \{x \in M_3(\mathbb{O}) \mid x = x^*\}$$
$$x \circ y = \frac{1}{2}(xy + yx)$$

In 1934 Albert proved that J_3^8 is not special.

Theorem (Jordan–von Neumann–Wigner 1934)

Any finite dimensional Euclidean Jordan algebra is a finite direct sum of simple, Euclidean, finite-dimensional Jordan algebras. Any finite-dimensional simple Euclidean Jordan algebra is isomorphic to one of:

$$\mathbb{R},\ JSpin_{n+2} = \mathbb{R} \oplus \mathbb{R}^{n+2}$$
$$J^1_{n+3},\ J^2_{n+3},\ J^4_{n+3},\ J^8_3 \qquad n \in \mathbb{N}.$$

Apart from J^8_3 all other simple finite dimensional euclidean Jordan algebras are special.

$Q = 2/3(Quarks)$	u	c	t
$Q = 0\ (Leptons)$	ν_e	ν_μ	ν_τ
$Q = -1/3(Quarks)$	d	s	b
$Q = -1(Leptons)$	e	μ	τ

In "*Exceptional quantum geometry and particle physics*" (May 2016), M. Dubois-Violette proposed a way to overcome the biggest flaw of Noncommutative Standard Model (quark-lepton symmetry and three generations of particles) allowing for quantum observables in the exceptional Jordan algebra J_3^8.

Consider the Hilbert space $\mathbb{C}^3$ equipped with the usual vector product $\times$. This product is nonassociative and $SU(3)-$invariant.
Equip $\mathbb{C}$ with the trivial representation of $SU(3)$

$$(g, z) \mapsto z \quad g \in SU(3),\ z \in \mathbb{C}.$$

Let $A = \mathbb{C} \oplus \mathbb{C}^3$, with the $SU(3)-$invariant product:

$$(z, Z)(z', Z') = (zz' - \langle Z, Z' \rangle, \bar{z}Z' - z'Z + iZ \times Z')\,.$$

Proposition

The algebra A is isomorphic to the algebra of octonions $\mathbb{O}$. $SU(3)$ is the subgroup of the special Lie group $Aut(\mathbb{O})$ which preserves the decomposition $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$.

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If we interpret $\mathbb{C}^3$ as the **internal colour space of quarks**, and $\mathbb{C}$ as the **(trivial) internal colour space of leptons** $\Rightarrow$ quark-lepton symmetry is just a consequence of $SU(3)-$colour symmetry.

From the decomposition $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ one gets $J_3^8 = J_3^2 \oplus M_3(\mathbb{C})$:

$$\begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_3 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & z_3 & \overline{z}_2 \\ \overline{z}_3 & \xi_2 & z_1 \\ z_2 & \overline{z}_3 & \xi_3 \end{pmatrix} \oplus (Z_1, Z_2, Z_3)$$

$$x_i = z_i + Z_i \quad x_i \in \mathbb{O}, z_i \in \mathbb{C}, Z_i \in \mathbb{C}^3, \xi_i \in \mathbb{R}.$$

Proposition

The subgroup of $Aut(J_3^8)$ which preserves the decomposition above is $(SU(3) \times SU(3))/\mathbb{Z}_3$, with action of $(U, V) \in (SU(3) \times SU(3))/\mathbb{Z}_3$:

$$H \mapsto VHV^* \;\; M \mapsto UMV^*$$

$$(H, M) \in J_3^2 \oplus M_3(\mathbb{C}).$$

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We have given the observables, now we should talk about states. Which leaves us with the question: **what does it mean to represent a Jordan algebra?**

Definition

Let J be a Jordan algebra, let M be a vector space equipped with a right and left bilinear maps:

$$
\begin{aligned}
J \otimes M \to M \quad & x \otimes \Phi \mapsto x\Phi \\
M \otimes J \to M \quad & \Phi \otimes x \mapsto \Phi x
\end{aligned}
$$

On $J \oplus M$, define the bilinear product $(x, \Phi)(x', \Phi') = (xx', x\Phi' + \Phi x')$, then M is a **Jordan bimodule** *if $J \oplus M$ endowed with this product is a Jordan algebra, that is:*

$$
\begin{gathered}
x\Phi = \Phi x \\
x(x^2\Phi) = x^2(x\Phi) \\
(x^2y)\Phi - x^2(y\Phi) = 2((xy)\Phi - x(y\Phi)) \\
\mathbb{1}_J\Phi = \Phi
\end{gathered}
$$

Example (Free modules)

Let J be a finite dimensional Jordan algebra. A free $J-$module M is:

$$M = J \otimes E$$

where E is a finite dimensional vector space and the action of J on M is given by multiplication on the first component of M.

As matter of fact every Jordan module over J_3^8 is a free module.

Now we get back to Standard Model

There are two familes for each generations $\Rightarrow$ take the module $M = J_3^8 \oplus J_3^8$, with the particle assignment:

$$J^u = \begin{pmatrix} \alpha_1 & \nu_\tau & \overline{\nu}_\mu \\ \overline{\nu}_\tau & \alpha_2 & \nu_e \\ \nu_\mu & \overline{\nu}_e & \alpha_3 \end{pmatrix} + (u, c, t) \quad J^d = \begin{pmatrix} \beta_1 & \tau & \overline{\mu} \\ \overline{\tau} & \beta_2 & e \\ \mu & \overline{e} & \beta_3 \end{pmatrix} + (d, s, b)$$

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$$H \mapsto VHV^* \ M \mapsto UMV^*$$
$$(H, M) \in J_3^2 \oplus M_3(\mathbb{C}).$$

- The action of $U \in SU(3)$ is responsible the usual color mixing.
- The action of $V \in SU(3)$ is responsible for the mixing of different generations of leptons.
- New real fields appears on the diagonal (There is room for a small step beyond Standard Model!).

Definition

For any category of algebras, the center of an algebra A is the commutative and associative subalgebra:

$$Z(A) = \{z \in A \mid [x, z] = 0, [x, y, z] = [x, z, y] = 0 \ \forall x, y \in A\}$$

where we have defined the **associator***:*

$$[x, y, z] = (xy)z - x(yz).$$

Definition

Let J be a Jordan algebra, let M and N be two modules over J, then a module homomorphism between M and N is a linear map $\varphi : M \to N$ such that

$$j\varphi(x) = \varphi(jx) \quad \forall\, x \in M, \forall\, j \in J.$$

We got the following result concerning homomorphism between free modules over Jordan algebras:

Proposition (A.C., L. Dabrowski, M. Dubois-Violette)

Let J be a finite dimensional, unital, simple Jordan algebra, let $M = J \otimes E$ and $N = J \otimes F$, where E, F are two finite dimensional vector spaces, be free modules over J. Then every module homomorphism $\varphi : M \to N$ is written as

$$\varphi(x \otimes v) = x \otimes Av \quad x \in J, v \in E$$

where $A : M \to N$ is a linear map.

Theorem

Let J be a finite dimensional, unital, Jordan algebra, let $M = J \otimes E$ and $N = J \otimes F$ be free modules over J, with E, F finite dimensional vector space of dimension m and n respectively. Then if $f : M \to N$ is homomorphism of J modules, there exist $\alpha_k \in Z(J)$ and $f_k \in M_{m\times n}$ such that:

$$f(1 \otimes e) = \sum_k \alpha_k \otimes f_k(e). \quad \forall e \in E.$$

In a more functorial fashion:

Theorem

Let J be a unital Jordan algebra with center $Z(J)$. Denote as $FMod_J$ the category of free Jordan modules over J with and as $FMod_{Z(J)}$ the category of free modules over the associative algebra $Z(J)$ with. Then the following functor is an isomorphism of categories:

$$\begin{aligned} \mathcal{F}: \quad & J \otimes E \to Z(J) \otimes E \\ & (\varphi : J \otimes E \to J \otimes F) \mapsto (\varphi_{Z(J)} : Z(J) \otimes E \to Z(J) \otimes F) \end{aligned}$$

where $\varphi_{Z(J)}$ is the restriction of φ to $Z(J) \otimes E$.

Definition

Let $\Omega = \oplus_{n\in\mathbb{N}}\Omega^n$ be a $\mathbb{N}$ unital graded algebra and for $x \in \Omega$, Ω is a **Jordan superalgebra** *if it is graded commutative, that is*

$$xy = (-1)^{|x||y|}yx$$

and respects the Jordan identity

One finds that Jordan identity is equivalent to:

$$(-1)^{|x||z|}\left[L_{xy}, L_z\right]_{gr} + (-1)^{|z||y|}\left[L_{zx}, L_y\right]_{gr} + \\ + (-1)^{|y||x|}\left[L_{yz}, L_x\right]_{gr} = 0. \qquad \forall x, y, z \in \Omega.$$

$Der(J) = \{X \in End(J) \mid X(xy) = X(x)y + xX(y)\}$ is a $Z(J)-$module and with the commutator as product it is a Lie algebra.

Definition

Let $\Omega^1(J)$ be $J-$ module of $Z(J)-$homomorphism of $Der(J)$ into J. Define the **differential***:*

$$d : J \to \Omega^1(J), \quad (dx)(X) = X(x) \quad \forall x \in J\ X \in Der(J).$$

We refer to $(\Omega^1(J), d)$ as **first order derivation based differential calculus** *over J.*

Then we extend d to a linear endomorphism of $\Omega_{Der}(J)$, on which d is an antiderivation and $d^2 = 0$. We refer to the pair (Ω, d) as **derivation based differential calculus.**

More generally:

Definition

If $\Omega = \oplus_{n\in\mathbb{N}}\Omega^n$ is a Jordan superalgebra and d is an antiderivation on Ω such that $d^2 = 0$, d is called a differential and Ω is called a **differential graded Jordan algebra***. If every Ω^n is a module over a Jordan algebra J we call (Ω, d) a* **differential calculus** *over J.*

More generally:

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If $\Omega = \oplus_{n \in \mathbb{N}} \Omega^n$ is a Jordan superalgebra and d is an antiderivation on Ω such that $d^2 = 0$, d is called a differential and Ω is called a **differential graded Jordan algebra**. *If every Ω^n is a module over a Jordan algebra J we call (Ω, d) a* **differential calculus** *over J.*

In general, derivation based differential calculus does not play any privileged role in the theory of differential calculus over a give Jordan algebra, with one interesting exception:

Theorem (A.C., L.Dabrowski, M. Dubois-Violette)

Let (Ω, d) be a differential calculus over J_3^8 and let $\phi : J_3^8 \to \Omega^0$ be an homomorphism of unital Jordan algebras. Then there exists a unique extension $\tilde{\phi} : \Omega_{Der}\left(J_3^8\right) \to \Omega$ such that $d \circ \tilde{\phi} = \tilde{\phi} \circ d_{Der}$.

Definition

A **connection** *on a Jordan module M is a linear map*

$$\nabla : Der(J) \to End(M)$$
$$X \mapsto \nabla_X$$

such that $\forall x \in J,\ m \in M$ *and* $z \in Z(J)$:

$$\begin{cases} \nabla_X(xm) = X(x)m + x\nabla_X(m) \\ \nabla_{zX}(m) = z\nabla_X(m) \end{cases}$$

It follows that $\nabla - \nabla' \in End(M)$.

Definition

The **curvature** *of a connection ∇ is*

$$R_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Example (Free modules)

$M = J \otimes E$, we have a base connection $\nabla^0 = d \otimes Id_E : J \otimes E \to \Omega^1_{Der} \otimes E$. Any connection on M is then written as:

$$\nabla = \nabla^0 + \mathcal{A}$$

$\mathcal{A} : Der(J) \to M_n(\mathbb{R})$.

We get the following characterization, which is very similar to its counterpart in the context of Lie algebra:

Proposition (A.C., L.Dabrowski, M. Dubois-Violette)

Flat connections on M are in one to one correspondence with Lie algebra homomorphisms $A : Der(J) \to M_n(\mathbb{R})$. That is, for a basis $\{X_\mu\} \subset Der(J)$ with structure constants $[X_\mu, X_\nu] = c^\tau_{\mu\nu} X_\tau$:

$$[A(X_\mu), A(X_\nu)] = c^\tau_{\mu\nu} A(X_\tau).$$

The study of connections on Jordan modules has revealed some interesting nonassociative geometry. These might be crucial in a future reformulation of standard model in this new context.
Some further studies:

- On the mathematical side:
  - Study Jordan module homomorphism and connections for non free modules.
  - Give a meaning to first order operators
- On the physical side:
  - Get $SU(2) \times U(1)$ gauge symmetry.
  - Study what happens when coupling with space-time degrees of freedom.
  - Write down a suitable action and study some dynamics.

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